Heisenberg's Uncertainty Relationships

Shortly after Schrödinger proposed his new equation, Heisenberg [7] developed his now famous relationships which would be used by Bohr to formulate the famous complementarity principle. The Heisenberg uncertainty principles for a quantum wave function are given in Equations (1.2):

$$\begin{aligned} \Delta x \Delta p &\geq \frac{\hbar}{2} \\ \Delta E \Delta t &\geq \frac{\hbar}{2}. \end{aligned} \tag{1.2}$$

As discussed further in Chapter 5, the time-energy uncertainty relation must be justified by a different method [8] than the position-momentum uncertainty because time does not have a well-defined quantum operator. Bohr also had a concise justification of the time-energy uncertainty relation in his Como paper where he first

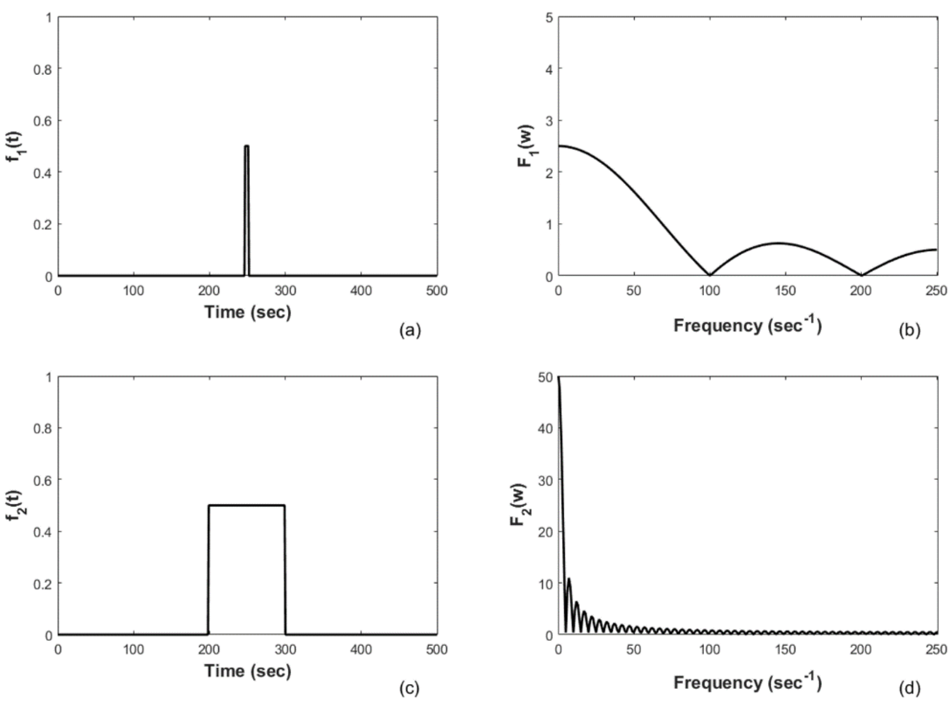

Figure 1.13: Functions of time and respective Fourier transforms: (a) time localized function $f_1(t)$; (b) Fourier transform of $f_1(t)$; (c) time uncertainty function $f_2(t)$; (d) Fourier transform of $f_2(t)$.

introduced complementarity which uses only the classical expressions for the resolving power of optical instruments [9, p. 312].

These uncertainty relationships are largely related to the well-known Fourier transform of a function $f(t)$, which we will denote as $F(w)$ where $w = 2\pi f$. Now one can see, from applying Einstein's result from the photoelectric effect $E = h\nu$, that w is also a measure of the energy of a photon. For example, Figure 1.13(a) shows a highly localized time function and Figure 1.13(b) the wide Fourier transform of the function. For the case of a more delocalized square-wave time function shown in Figure 1.13(c), the Fourier transform in Figure 1.13(d) is significantly more localized in frequency when compared to Figure 1.13(b). As one further localizes a function in time, the energy becomes less certain. Variables such as time and energy that are related in this manner are called conjugate variables and are important for understanding Bohr's complementarity argument. In terms of the issue of measurement, Bohr would argue that the interaction with the device necessarily would have an effect on the particle's properties due to the Heisenberg uncertainty principle. Such a change would be in principle uncontrollable. One can consider that if a measurement device were to interact with a particle in such a manner that the particle's time when it entered the device could be known very precisely, then the particle's energy would have to have a substantial uncertainty. The gray areas shown in Figure 1.14 are valid time-energy uncertainty products, while other such products would violate the Heisenberg uncertainty principle, particularly where both the time uncertainty and energy uncertainty are small.

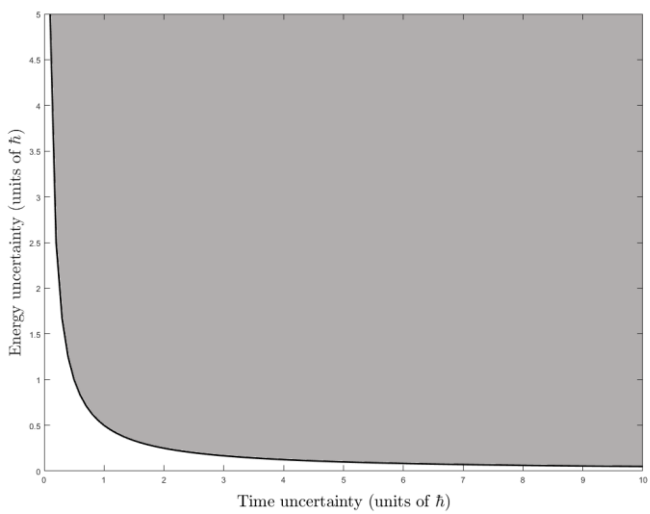

Figure 1.14: Energy versus time uncertainties as a consequence of Heisenberg's uncertainty relationships. Valid energy-time uncertainties shown in gray.

This change in uncertainty for Bohr is applicable to the inevitable consequence of requiring an interaction between the particle and device in order for the device to register a measurable quantity, such as the particle's time-of-arrival.

Consider the wave-particle light pulse in its interaction with a measurement device that will measure the position of the particle. If the position of the particle is very accurately determined, its momentum must be spread or uncertain. Or if the measurement device is such that the device can ascertain the momentum of the particle accurately, then the particle's position becomes uncertain. Now when the momentum becomes certain for a photon, then the position becomes uncertain and a plane wave results. One could then ask whether each particle has an in-principle unknowable position, but nonetheless does have an absolute position. This is an example of Einstein's objections with quantum mechanics in his debates with Bohr.