Mathematics of Quantum Entanglement

Consider the state ρ_{AB} of a composite system comprising just two subsystems A and B. The simplest case is a product state, $|\psi_{AB}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$ or $\rho_{AB} = |\psi_A\rangle\langle\psi_A| \otimes |\psi_B\rangle\langle\psi_B|$, which is separable or *non-entangled.* Note that in this section, the states are described explicitly using the notation of tensor products. The possibility of entanglement enters because quantum states can also be in superpositions. A simple example of a quantum entangled state is the superposition state of two qubits A and B

$$|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}(|0_A\rangle \otimes |0_B\rangle + |1_A\rangle \otimes |1_B\rangle) \tag{2.17}$$

$$\begin{aligned} \rho_{AB} &= |\psi_{AB}\rangle\langle\psi_{AB}| \\ &= \frac{1}{2}(|0_A\rangle\langle 0_A| \otimes |0_B\rangle\langle 0_B| + |1_A\rangle\langle 1_A| \otimes |1_B\rangle\langle 1_B| \\ &+ |0_A\rangle\langle 1_A| \otimes |0_B\rangle\langle 1_B| + |1_A\rangle\langle 0_A| \otimes |1_B\rangle\langle 0_B|). \end{aligned} \tag{2.18}$$

The entanglement enters from the cross-terms $|0_A\rangle\langle 1_A| \otimes |0_B\rangle\langle 1_B| + |1_A\rangle\langle 0_A| \otimes |1_B\rangle\langle 0_B|$. Otherwise the state would be separable. More generally, a separable state takes the form of a mixture of product states and it is *entangled* if it *cannot* be written as the separable mixture of product states of the form [42]

$$\rho_{AB}^{sep} = \sum_k p_k \rho_A^k \otimes \rho_B^k = \sum_k p_k |\psi_A^k\rangle\langle\psi_A^k| \otimes |\psi_B^k\rangle\langle\psi_B^k|. \tag{2.19}$$

where $\sum_k p_k = 1$. The physical meaning of this is that it should not be possible to create entangled states just by acting on one of the subsystems individually. For example, if *local* unitaries U_A and U_B separately act on A and B respectively, no entanglement can be created,

$$\acute{\rho} = (U_A \otimes U_B)\rho(U_A \otimes U_B)^\dagger. \tag{2.20}$$

Note that the concept of a state being entangled is defined in terms of what it is not, namely Equation (2.19). This is due to the difficulty of specifying the most general entangled state. However, entanglement measures have been developed that can be applied to a range of quantum states. There are several criteria for a proper measure $E[\rho]$ for quantifying the amount of entanglement in a quantum state and several different measures have been proposed [43]. A proper measure of quantum entanglement should fulfill the following requirements [44]:

(E1) For any separable state, $E[\rho] = 0$.

(E2) The entanglement remains unchanged under local unitaries, $E[(U_A \otimes U_B)\rho(U_A \otimes U_B)^\dagger] = E[\rho]$.

(E3) Local operations and classical communication (LOCC) cannot increase the expected entanglement.

(E4) Entanglement of a pure state is equal to the von Neumann entropy of the subsystem state.

As an example, consider the two-qubit entangled pure state Equation (2.17) with density operator $\rho_{AB} = |\psi_{AB}\rangle\langle\psi_{AB}|$ and use criterion (E4) to evaluate its entanglement. The subsystem state is a simple diagonal matrix with von Neumann entropy evaluated using Equation (2.38) as (Exercise 2.3 in the book or kindle version of theQMP))

$$S_A = -\mathrm{Tr}(\rho_A \log \rho_A) = \log 2. \tag{2.21}$$

Therefore, the state in Equation (2.17) is maximally entangled with $E[\rho_{AB}] = S_A = \log 2$. In the literature, the von Neumann entropy is sometimes defined using the logarithm to base 2 instead, so that the maximum entanglement for two qubits is $E[\rho_{AB}] = 1$. Another entanglement measure for two qubits is the *concurrence*, which is a proper measure for both pure and mixed states. This is discussed and used in Chapter 3 as part of the test for distinguishing unitary from non-unitary processes.