Quantum Evolution with Subsystem Interactions

In the presence of interactions between subsystems, when the degrees of freedom of some of the subsystems are traced out, the evolution of the remaining set of subsystems generally will not remain unitary, but instead becomes described by a non-unitary stochastic equation. This is the problem of open systems, which is discussed further in Chapter 7. It occurs, for example, when the system of interest may be coupled to an environment that cannot be easily described in detail. When the degrees of freedom of the environment are traced out, the unitary evolution becomes modified by stochastic terms representing the influence of the environment. An open quantum system suffers noise and decoherence due to interaction with its surroundings. This leads to a type of evolution $\mathcal{E}(\rho)$ more general than unitary evolution. The requirements on $\mathcal{E}(\rho)$ are that it be trace-preserving (since we must have $\mathrm{Tr}\rho = 1$) and *completely positive* (CP), meaning that it remains positive even when we extend the map as $\mathcal{E}\otimes I$ and apply it to a larger composite system by appending an *ancilla* environment E to our quantum system. This is required because the quantum system of interest must interact with other physical systems as necessary for a complete description of the world, and the evolution of the interacting system must still remain physically valid. However, even these more general maps $\mathcal{E}(\rho)$ can be described by unitary transformations in terms of an ancilla environment as long as it is not initially correlated with the quantum system. This is a result of *dilation theorems* due to Naimark [46] and Stinespring [47] [48]. If E is initially in the state $|0\rangle$, a unitary operator U on the composite system can always be found so that the CP map $\mathcal{E}(\rho)$ takes the following form for all ρ,

$$\mathcal{E}(\rho) = \mathrm{Tr}_E U(\rho\otimes|0\rangle\langle 0|)U^{\dagger}. \tag{2.29}$$

The devotion to describing irreversible processes in terms of reversible unitary operators of a larger system has been called going to the Church of the Higher Hilbert Space. One of the questions of the Quantum Measurement Problem is whether or not Nature is a devotee to this creed and is always able take refuge in the Church of the Higher Hilbert Space.

A more general description of CP maps is the *operator-sum representation* [49]. There exist operators A_μ such that

$$\mathcal{E}(\rho) = \sum_\mu A_\mu \rho A_\mu^{\dagger}. \tag{2.30}$$

where $\sum_\mu A_\mu^{\dagger} A_\mu = 1$. The operators A_μ are generally not unitary and their representations are not unique. An important property of quantum evolutions is that they cannot be used to signal instantaneously. If Alice and Bob are spatially separated, then a measurement by Alice cannot immediately influence Bob’s state. This can be demonstrated by taking the partial trace of the overall system with respect to Alice’s system, whereby it can be shown (Exercise 2.6 in the book or kindle version of theQMP) via a direct calculation,

$$\mathrm{Tr}_A\big(\mathcal{E}(\rho)\big) = \mathrm{Tr}_\mathrm{A}\left(\sum\nolimits_\mu \big(A_\mu \otimes I_B\big)\rho\big(A_\mu \otimes I_B\big)^\dagger\right) = \mathrm{Tr}_A(\rho). \qquad (2.31)$$

Equation (2.31) implies that Bob cannot distinguish a random measurement from any actions taken by Alice. Therefore, quantum mechanics is protected against the unphysical occurrence of instantaneous signaling.