Subsystem Entropies and the Araki-Lieb Inequality

The *Araki-Lieb Inequality* [50] is an important result of quantum correlations. Let the composite system of A and B be given by the state ρ_{AB} and the reduced density matrices of the two sub-systems A and B be ρ_A and ρ_B. Then the total entropy and subsystems entropies are related by

$$S(\rho_A) + S(\rho_B) \geq S(\rho_{AB}) \geq |S(\rho_A) - S(\rho_B)|. \tag{2.35}$$

Note that the entropy of the total system $S(\rho_{AB})$ can be less than that of A and B considered separately. In this case, there is more information in the total system due to correlations between A and B, which may include the effect of quantum entanglement. If ρ_{AB} is a pure state, then $S(\rho_{AB}) = 0$ so that $S(\rho_A) = S(\rho_B)$ from the right-hand side of Inequality (2.35). For a bipartite pure state, the subsystem entropies are always equal no matter how disparate the two physical systems. An example is an atom in an electromagnetic field. Remarkably, as long as the combined system is in a pure state, the finite degrees of freedom of the atom and the infinite degrees of freedom of the field each must have identical entropies. Interactions between subsystems A and B result in correlations with an additional measure of information called their mutual information $I(A{:}B)$ [44], the information about A gained by learning about B or vice-versa,

$$I(A{:}B) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}). \tag{2.36}$$

In the special case where A and B are independent, the entropies of the subsystems simply add, as is also the case for classical systems.

$$S(\rho_A) + S(\rho_B) = S(\rho_A \otimes \rho_B). \tag{2.37}$$

As an example, consider the two-qubit entangled pure state Equation (2.17) with density operator $\rho_{AB} = |\psi_{AB}\rangle\langle\psi_{AB}|$ and use criterion (E4) to evaluate its entanglement. The von Neumann entropy of a pure state is zero so that $S(\rho_{AB}) = 0$. However, the entanglement of ρ_{AB} is given by the von Neumann entropy of each of the subsystems ρ_A and ρ_B. The entropies of these are identical as we know from the Araki-Lieb Inequality (2.35), $S(\rho_A) = S(\rho_B)$, so we only need to calculate the reduced state for one of the subsystems, for example (Exercise 2.7 in the book or kindle version of theQMP)

$$\rho_A = \mathrm{Tr}_B(\rho_{AB}) = \frac{I_2}{2}. \tag{2.38}$$

where I_2 is the 2x2 identity matrix. This indicates that, although the total state ρ_{AB} is pure, each of the subsystems is a *maximally mixed* or chaotic state. The state in Equation (2.17) is maximally entangled with $E[\rho_{AB}] = S_A = \log 2$. Therefore, it has the maximum possible entropy and maximum entanglement.