Theorems of Wigner and Stone

Eugene Wigner [12] [45, p. 71] proved that all operations in quantum mechanics corresponding to symmetries must be either unitary U (e.g., rotations, translations, reflections) or anti-unitary UK (e.g., time-reversal), where K denotes complex conjugation and that these preserve probabilities. Therefore, the new states after a symmetry operation will obey the same laws of nature as the original states. The symmetry of time-translation corresponds to unitary time-evolution. *Wigner's Theorem* allows us to speak as if the pure states $|\psi\rangle$ of quantum mechanics live in Hilbert space. Technically, it is the entire set $\{e^{i\alpha}|\psi\rangle\}$ (called a *ray*) for all real phases α that describes one physical state since the phases do not affect probabilities or expectation values. However, Wigner's result means that all mappings between rays that preserve probabilities can be accomplished by either unitary or anti-unitary mappings between vectors in Hilbert space. That most of the physically significant transformations in quantum mechanics are unitary is due to Wigner's theorem.

A particular system and its interactions will determine the specific form for U. This can often be facilitated by *Stone's Theorem* [11] [37, p. 264] which implies that for every (strongly continuous) unitary operator U, there is a Hermitian operator H (so that $H = H^\dagger$), called the Hamiltonian, such that

$$U(t) = \exp(-iHt/\hbar). \tag{2.22}$$

The Hamiltonian represents the energies of the system and its interactions and the unitary evolution is determined by these energies. This implements the time-translation invariance symmetry of Wigner's theorem—the laws of nature do not depend on how our clocks are set. Operating with the unitary operator of Equation (2.22) simply shifts the time (Exercise 2.1)

$$U(\tau)|\psi(t)\rangle = |\psi(t+\tau)\rangle. \tag{2.23}$$

The time translation symmetry corresponds to conservation of energy, since $UHU^\dagger = H$. Similarly, if the laws of nature are independent of a shift in spatial translation (i.e. where we mark the origin of our coordinate system), there must exist an equivalent unitary operation. This corresponds to conservation of linear momentum, with the spatial translation unitary operator given by

$$U_P(\bar{x}) = \exp(-i\bar{P}\cdot\bar{x}), \tag{2.24}$$

and which shifts the spatial location according to (Exercise 2.1):

$$U_P(\bar{a})|\psi(\bar{x},t)\rangle = |\psi(\bar{x}+\bar{a},t)\rangle. \tag{2.25}$$

For each symmetry, there is a unitary operator to generate the corresponding transformation in terms of the complementary canonical coordinate [45, p. 69]. Symmetries can be *continuous* (rotations, translations) or *discrete* (time reversal, space-inversion or parity). The unitary operators for continuous symmetries differ infinitesimally from the identity by a Hermitian operator G called the *generator* of the corresponding symmetry

$$U = I - i\frac{\varepsilon}{\hbar}G + O(\varepsilon^2). \tag{2.26}$$

If the symmetry is conserved by the time evolution so that $UHU^\dagger = H$, then we can conclude from Equation (2.26) that:

$$[G, H] = 0 \tag{2.27}$$

where the *commutator* is given by $[G, H] \equiv GH - HG$. Therefore, conservation laws can be related to commutation relations of the Hamiltonian (Exercise 2.5).

In the Heisenberg representation, the time-dependence is unitarily transferred from the Schrödinger representation wave function $|\psi(t)\rangle$ to the operators of the problem such as G, which obeys

$$\frac{dG}{dt} = -\frac{i}{\hbar}[G, H]. \tag{2.28}$$

If G is a constant of the motion, then we can conclude from Equations (2.27) and (2.28) that $dG/dt = 0$. For example, if H is invariant under spatial translation, then $[P, H] = 0$ and, therefore, the momentum P is a constant of the motion, $dP/dt = 0$.