Unitary Interference Operations

Consider a simple unitary transformation which rotates any single qubit state by the angle θ using the basis of the two orthogonal states, $|0\rangle$ and $|1\rangle$ and written as the following 2x2 matrix (Exercise 2.2):

$$U_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{2.13}$$

For $\theta = 45$ degrees, $\sin\theta = \cos\theta = 1 / \sqrt{2}$, and this will rotate the original basis into another orthonormal basis given by the coherent superpositions of $|0\rangle$ and $|1\rangle$: $|+\rangle \equiv (|0\rangle + |1\rangle)/\sqrt{2}$ and $|-\rangle \equiv (|0\rangle - |1\rangle)/\sqrt{2}$,

$$U_\theta|0\rangle = |-\rangle = (|0\rangle - |1\rangle)/\sqrt{2}\,,\; U_\theta|1\rangle = |+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}. \tag{2.14}$$

Thus, unitaries can create coherent superpositions between states. But they can also destroy coherence by destructive interference. Applying U_θ once more to Equation (2.14) cancels the coherence:

$$U_\theta U_\theta|0\rangle = -|1\rangle, U_\theta U_\theta|1\rangle = |0\rangle. \tag{2.15}$$

In fact, a unitary operator can be viewed as simply equivalent to a change of basis, since if $\{|k\rangle\}$ is an orthonormal basis, then so is $\{U|k\rangle\}$, because $(U|l\rangle)^\dagger U|k\rangle = \langle l|U^\dagger U|k\rangle = \langle l|k\rangle = \delta_{lk}$, which is the requirement for orthonormality.

For a single qubit, the most general unitary operator can be written in terms of the Pauli matrices σ_i as

$$U_\theta = (\cos\theta\, I \;+ i\sin\theta\;\hat{n}\cdot\bar{\sigma}\,). \tag{2.16}$$

This represents a rotation by angle θ around the direction of the unit vector $\hat{n}$ and generalizes Equation (2.13) as a convenient way for unitary manipulation of photon polarization modes.

The probability distribution for interference experiments such as the double-slit experiment is not simply the sum of the two distributions measured when only one slit is open since there is a third term from the interference between the two. Sorkin [40] has demonstrated a remarkable property of quantum mechanics that follows from this if we consider the more complicated situations of three or more slits. He showed that the probability distribution for multiple slits can always be written as sums of the possible cases where only one or two of the slits are open so that there is no interference exclusive to having more than two slits. In other words, quantum superposition and Born's rule imply that quantum interference always occurs from pairs of possible paths through the slits, and any deviation from this would indicate a failure of Born's rule [41]. The measured properties of quantum mechanics all stem from the interference of pairs of possibilities.