Bell's Inequality

The theory of entanglement is paramount in the development and understanding of the measurement problem. As will be further discussed in Chapter 4, the problem of the unitary prediction of entanglement versus the expectation that such entanglement is not present in local classical physics, was understood by Einstein rather earlier on historically than is generally realized. With the discovery by Bell [81] in 1964 of inequalities that are satisfied by any local hidden variable model and yet violated by quantum mechanical predictions, the issues of entanglement and nonlocality became clear. Experiments beginning in 1972 and refined over subsequent years by Clauser et al. [82], Aspect et al. [83], Ou et al. [84], and Shih et al. [85] ultimately led to the larger physics community's acceptance of entanglement and later, with proposed applications and theory of entanglement, to the emergence of the field of Quantum Information. The remaining loopholes that have remained regarding nonlocality due to detector inefficiency, etc., have recently been essentially closed [86] [87].

One might have expected then, with entanglement being understood and accepted by many, that the Quantum Measurement Problem, which deals with the dichotomy between entanglement and separable systems would be clearly understood and appreciated by most. This is not the case—the great majority of experiments (there are exceptions but often these systems violate the continuous Gaussian bound rather than Bell's inequality) in quantum information deal with the dichotomy between the entanglement of very few particles versus separable systems. On the other hand, the measurement problem requires one to confront the issue of the entanglement of detectors which are often considered to be macroscopic bodies.

The two particles that are emitted impinge on (separate) beam splitters. In the case when the source emits photons that are entangled in polarization, the beam splitter is assumed to be a polarizing beam splitter. After the polarizing beam splitter there are two detectors A and B corresponding to the two possible non-reflected and reflected outputs of the polarizing beam splitter. If the photon emitted toward Detector A is detected in the reflected path, A is assigned the value -1, otherwise A=1. Similarly, if the photon emitted toward Detector B is detected in the reflected path, B is assigned the value -1, otherwise B=1.

Each polarizing beam splitter can be set at an arbitrary angle θ that would allow an input horizontal polarization state $|1_H\rangle$ to always pass through un-reflected when $\theta = 0$ but has an increasing probability of being reflected as θ is increased until an incoming $|1_H\rangle$ is completely reflected when $\theta = 90°$.

Over multiple trials of the experiment, the statistic $\hat{C}$ is defined as the average of the product of the outcomes of A and B and is a measure of the statistical correlation between random variables A and B. For a given set of beam splitter parameters θ_A, θ_B the correlation computed is denoted $\hat{C}(\theta_A, \theta_B)$.

When the Source S emits particles, one can consider the possibility of a hidden variable parameter defined by $\lambda \in \mathbb{R}$ where $\mathbb{R}$ denotes the real numbers such that the detection result at A is a function of only λ and θ_A while the detection result at B is a function of only λ and θ_B. In this case note that $A(\lambda, \theta_A)$ cannot be a function of the parameter setting θ_B. Such a model is defined to be a local hidden variable (LHV) model. The Clauser-Horne-Shimony-Holt (CHSH) inequality [88] is a more widely used extension of Bell's original inequality [81] that is easier to test experimentally. The CHSH inequality consists of computing the sample $\hat{C}(\theta_A, \theta_B)$ for four sets of experimental configurations with parameters $\{(\theta_A, \theta_B), (\acute{\theta}_A, \theta_B), (\theta_A, \acute{\theta}_B), (\acute{\theta}_A, \acute{\theta}_B)\}$.

For each experimental configuration, one computes $\hat{C}$ over the series of experimental trials. As shown in [88], the CHSH inequality provides that any LHV model will satisfy

$$S = \left|\hat{C}(\theta_A, \theta_B) + \hat{C}(\acute{\theta}_A, \theta_B) + \hat{C}(\theta_A, \acute{\theta}_B) - \hat{C}(\acute{\theta}_A, \acute{\theta}_B)\right| \leq 2. \tag{3.2}$$

This inequality is also satisfied by every separable quantum state. However, it may be violated by any pure entangled quantum state up to a maximum possible violation given by $S = 2\sqrt{2}$, known as the Cirel'son bound [89]. Violation of the local CHSH inequality is a sufficient condition for detection of entanglement between the two parties, regardless of the system and details of implementation. For every pure entangled state, a set of local measurements can be found so that the resulting correlations will violate the CHSH inequality [90] [91]. For mixed states, the relation between entanglement and nonlocality is subtler since any associated nonlocality sometimes cannot be revealed using only direct measurements [92].

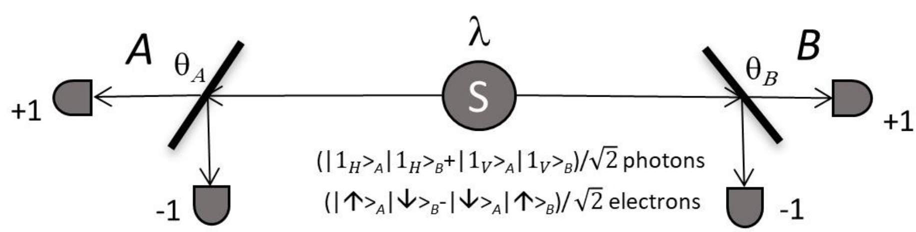

Figure 3.3 Experimental configuration for testing Bell's Inequality.

As an example of a violation of the CHSH inequality at the Cirel'son bound, consider the maximally entangled input state $(|1_H\rangle_A|1_H\rangle_B + |1_V\rangle_A|1_V\rangle_B)/\sqrt{2}$. The quantum mechanical prediction can be shown to be given by $C(\theta_A, \theta_B) = \cos(2(\theta_A - \theta_B))$ where we define $C(\theta_A, \theta_B) \equiv E(\hat{C}(\theta_A, \theta_B))$ and denote the expected value of a random variable x by $E(x)$. Then this maximally entangled input state and the particular measurement settings separated by successive 22.5-degree angles given by

$$\{(0°, 22.5°), (0°, 67.5°), (45°, 22.5°), (45°, 67.5°)\} \tag{3.3}$$

produce a correlation at Cirel'son's bound $S = 2\sqrt{2} \approx 2.8284$, which significantly violates Equation (3.2). There have now been numerous experimental confirmations of violations of Bell's inequality, closing existing "loopholes" provided by additional assumptions that previous experimenters were forced to make due to technological constraints (such as low detection efficiency for which hidden variable models still exist, even though the correlation S exceeds Bell's inequality) [93] [86] [94]. The preponderance of evidence indicates that Nature is non-local and consistent with quantum mechanics.

Separable quantum states are restricted by the CHSH bound $S_{\text{sep}} \leq 2$ consistent with LHV models while entangled quantum states can achieve the larger Cirel'son bound $S_{\text{ent}} \leq 2\sqrt{2}$. In order to achieve the maximal violation for entangled quantum states, both pairs of local observables must be chosen to be anti-commuting [91]. Remarkably, the case of anti-commuting observables also affects the case of separable quantum states, resulting in a strengthened CHSH bound of $S_{\text{sep}} \leq \sqrt{2}$, considerably smaller than the CHSH bound of $S_{\text{sep}} \leq 2$ [95] [96]. As will be seen in this chapter, the bound of $\sqrt{2}$ can be achieved by pure product states for which there is no quantum entanglement. The dichotomy between unitary and non-unitary processes can be quantified in terms of the contrast between the corresponding limits of $S_{\text{product}} = \sqrt{2}$ and $S_{\text{entangled}} = 2\sqrt{2}$ in Bell measurements.