Extension to Mixed States

In the prior development, it was assumed that the initial ready state of Device i is the pure state $|\psi_r^0\rangle_i$, *i=1,2*, $|\psi_r^0\rangle_i \in \mathcal{H}_i$, *i=1,2*, $\mathcal{D}(\mathcal{H}_i) = N$. The joint state of Device 1 and 2 was assumed to be a tensor product state given by $|\psi_r^0\rangle_1 \otimes |\psi_r^0\rangle_2$. The UMDT procedure developed previously is herein extended to the case of mixed states.

The initial ready state of Device i will be assumed to be described by the density matrix $\rho_{r,i}^0$, $\mathcal{D}(\mathcal{H}_i) = N$. Given a Hilbert space $\mathcal{H}$, a density matrix ρ sometimes referred to as a density operator $\rho: \mathcal{H} \to \mathcal{H}$, is a bounded positive semi-definite Hermitian unity trace operator. The set of density matrices, denoted $\mathfrak{M}(\mathcal{H})$, is a convex set with extremal elements that are pure states.

A Hermitian positive semi-definite matrix $\rho_{r,i}^0 \in \mathfrak{M}(\mathcal{H})$ can be written as

$$\rho_{r,i}^0 = \sum_{j=1}^{R_i} \lambda_j^i |\psi_{r,j}^0\rangle_{i\,i}\langle \psi_{r,j}^0|$$

where $|\psi_{r,j}^0\rangle_i \in \mathcal{H}_i$,*i=1,2* are any set of orthogonal eigenstates of the density matrix with respective eigenstates $\lambda_j^i \geq 0$. The rank R_i of the density matrix denoted $\mathcal{R}\left(\rho_{r,i}^0\right)$ is equal to unity in the case that the state is pure and is strictly greater than 1 in the case that the matrix is mixed.

The initial photon state after the PBS will be assumed as before to be a pure state given as

$$|\psi_{\text{photon,PBS}}\rangle = \sqrt{a}|1_V\rangle_B \otimes |0\rangle_C + \sqrt{1-a}|0\rangle_B \otimes |1_H\rangle_C,$$

where $a \in \mathbb{C}$, $|a| \leq 1$ is the degree of superposition. The density operator of the photon initial state is represented as

$$\rho_{\text{photon,PBS}} = |\psi_{\text{photon,PBS}}\rangle \langle\psi_{\text{photon,PBS}}|.$$

The initial state of the Ancilla, Photon, Device 1, and Device 2 can be written given as

$$\tilde{\rho}_{\text{init}} = \rho_{A_1,A_2}^0 \otimes \rho_{\text{photon,PBS}} \otimes \rho_{r,1}^0 \otimes \rho_{r,2}^0. \tag{3.29}$$

where $\rho_{A_1,A_2}^0 = \rho_{A1}^0 \otimes \rho_{A2}^0$ is the initial joint density matrix of Ancilla 1 and 2. This can be re-written in a manner similar to the case of pure states whereby Ancilla 1 and the B photon modes are adjacent to Device 1 while Ancilla 2 and C photon modes have been inserted adjacent to Device 2, which will be denoted by ρ_{init}. As in the case of the pure state treatment, the composite system of the joint state of the photon in the B port and Device 1 will be referred to as System $1'$ while System $2'$ denotes the joint state of the photon in the C port and Device 2. Let $U_{\text{sw}}^{(1)}$ denote the unitary operator that swaps Mode C with Device 1 so that System $1'$ and $2'$ are adjacent and denote

$U_{\text{sw}}^{(2)}$ the unitary operator that changes the position of Ancilla 2 so that it is inserted between System $1'$ and $2'$. In this case,

$$\rho_{init} = U_{\text{sw}}^{(3)} \tilde{\rho}_{\text{init}} U_{\text{sw}}^{(3)\,\prime}$$

where $U_{\text{sw}}^{(3)}$=$U_{\text{sw}}^{(2)} U_{\text{sw}}^{(1)}$. Consider the unitary evolution of the initial state of Equation (3.29) followed by the Step 1 unitary evolution to be specified via the matrices X and Y, which is then followed by the Step 2 Bell experiment which consists of measuring a set of observables. The expected value one of any of the observables $\mathcal{O}$ is given by,

$$\mathcal{E}(\rho_{\text{init}}, \mathcal{O}) = \text{Tr}(\mathcal{O} U_c (\rho_{A_1,A_2}^{(0)} \otimes \rho_{\text{photon,PBS}} \otimes \sum_{j=1}^{R_1} \lambda_j^i |\psi_{r,j}^0\rangle_1 {}_1\langle \psi_{r,j}^0| \otimes \rho_{r,2}^0) U_c{}')$$

where $U_c \equiv (X \otimes Y) U_{\text{sw}}^{(3)}$. Denoting the initial density matrix

$$\rho_{\text{init}}(j) \equiv \rho_{\text{photon,PBS}} \otimes |\psi_{r,j}^0\rangle_1 {}_1\langle \psi_{r,j}^0| \otimes \rho_{r,2}^0,$$

it is found by completing Exercises 3.4-3.6 (in the book or kindle version of theQMP) that $\mathcal{E}\big(\rho_{\text{init,}} \mathcal{O}\big)$ is completely characterized by $\mathcal{E}(\rho_{\text{init}}(j), \mathcal{O})$, $\forall$ j.

As we already have developed a methodology of specifying T_1 when both Device 1 and 2 are each pure states, let us for now hold j constant and assume that Device 1 is initialized to $|\psi_{r,j}^0\rangle_1 {}_1\langle \psi_{r,j}^0|$ and Device 2 initialized to a pure state. Consider then the following extension to the methodology of specifying T_1 for mixed states, for which after Step 1, $\mathcal{E}(\rho_{\text{init}}(j), \mathcal{O})$ has transferred the entanglement that existed between System $1'$ and System $2'$ to entanglement of the two qubits. The theory will be completed if a linear operator $T_1 \otimes T_2$ can be specified that achieves the desired entanglement transfer between System $1'$ and System $2'$ into entanglement of the two ancilla, for *all* of Device 1's eigenstates $|\psi_{r,j}^0\rangle_1$. With this goal in mind, define

$$|e_{1,j,}^0\rangle_{1'} = |1_V\rangle_B \otimes |\psi_{r,j}^0\rangle_1$$
$$|e_{2,j}^0\rangle_{1'} = |0\rangle_B \otimes |\psi_{r,j}^0\rangle_1$$

and $\big|e_{i,j}^1\rangle_{1'} = W\big|e_{i,j}^0\rangle_{1'}$, $i = 1{,}2$ as the unitarily evolved states after interaction of Photon B mode with Device 1. The solution for T_1 when Device 1 is initially in a pure state has previously been found and can be written (with a slight change to the notation)

$$T_{1,j} \equiv (|1\rangle_{A_1} \otimes |e_{1,j}^1\rangle_{1'})(\langle 0|_{A_1} \otimes {}_{1'}\langle e_{1,j}^1|) + (|0\rangle_{A_1} \otimes |e_{1,j}^1\rangle_{1'})(\langle 0|_{A_1} \otimes {}_{1'}\langle e_{2,j}^1|).$$

A linear map is needed that maps $|0\rangle_{A_1} \otimes |e_{1,j}^1\rangle_{1'}$ to $|1\rangle_{A_1} \otimes |e_{1,j}^1\rangle_{1'}$ and $|0\rangle_{A_1} \otimes |e_{2,j}^1\rangle_{1'}$ to $|0\rangle_{A_1} \otimes |e_{1,j}^1\rangle_{1'}$. Hence the mapping for the case of pure states must be extended to hold not only for any single *j*, but for all *j*. Consider the ansatz that the form of T_1 is the sum of these $T_{1,j}$, i.e.,

$$T_1 = \sum_{j=1}^{R} T_{1,j}. \tag{3.30}$$

Note that T_1 is a linear operator because the sum of linear operators is itself a linear operator. Given a similar construction for T_2, $T_1 \otimes T_2$ achieves the desired transfer from the entanglement between System $1'$ and System $2'$ to entanglement between the two Ancilla, see Exercise 3.8-3.9 in the book or kindle version of theQMP. Furthermore, T_1 and T_2 can be extended to unitary mappings X and Y, see Exercises 3.10-3.12 in the book or kindle version of theQMP.

So far, a procedure has been developed to handle any initial state that is of the form whereby one of the two Devices is initialized to a mixed state while the other device is initialized to a pure state. To complete the argument, the latter result is extended in which Device 1 and Device 2 can both be initialized to arbitrary mixed states, see Exercise 3.13 in the book or kindle version of theQMP. Hence, we have specified a UMDT for arbitrary initial device states.