Geometry of Entanglement

Consider an arbitrary product pure state where $|\psi\rangle = |e_1\rangle_1 \otimes |f_1\rangle_2$. This is a single product state hence by Theorem 3.1, has Schmidt rank 1. Consider the geometry of this state represented by two vectors representing the kets $|e_1\rangle_1$ and $|f_1\rangle_2$ illustrated in 3-dimensional space for visualization purposes in Figure 3.5.

![Figure 3.5: Geometric visualization of a product bipartite state with Schmidt vectors $|e_1\rangle_1$ and $|f_1\rangle_2$.](image)

This can also be looked at as a single vector in the joint Hilbert space composing System 1-System 2, $\mathcal{H}_{1,2}$ as shown in Figure 3.6. In reality, the vector will exist in a higher than 3-dimensional space.

In an entangled system of particles in a pure state, the quantum state cannot be decomposed as a single product state. In Figure 3.7 an entangled bipartite system is illustrated that is composed of the superposition of two product states, i.e.,
\[ |\psi\rangle = \sqrt{a}|e_1\rangle_1 \otimes |f_1\rangle_2 + \sqrt{1-a}|e_2\rangle_1 \otimes |f_2\rangle_2 \] where \( a \in \mathbb{C}, |a| \leq 1 \).

The dotted line between the vectors is shown to be weighted by the coefficients of each respective product state. There is a 90° angle between the orthogonal states \(|e_1\rangle_1\) and \(|e_2\rangle_1\) and similarly between \(|f_1\rangle_2\) and \(|f_2\rangle_2\). This entangled state can also be visualized as a superposition of \(|\psi_{PS,1}\rangle = |e_1\rangle_1 \otimes |f_1\rangle_2\) and \(|\psi_{PS,2}\rangle = |e_2\rangle_1 \otimes |f_2\rangle_2\), where the subscript PS denotes product state. In this case, \(|\psi\rangle = \sqrt{a}|\psi_{PS,1}\rangle + \sqrt{1-a}|\psi_{PS,2}\rangle\).

This is illustrated in Figure 3.8. The degree of the superposition \(\sqrt{a}\) related to the entanglement, can be seen to simply correspond geometrically to the angle between the entangled state \(|\psi\rangle\) and the product state \(|\psi_{PS,1}\rangle\) via \(\theta = \arccos(\sqrt{a})\). The three vectors \(|\psi_{PS,1}\rangle, |\psi_{PS,2}\rangle\), and \(|\psi\rangle\) can be seen to all lie in a single plane determined by any two of these vectors.