Geometry of Entanglement

Consider an arbitrary product pure state where $|\psi\rangle = |e_1\rangle_1 \otimes |f_1\rangle_2$. This is a single product state hence by Theorem 3.1, has Schmidt rank 1. Consider the geometry of this state represented by two vectors representing the kets $|e_1\rangle_1$ and $|f_1\rangle_2$ illustrated in 3-dimensional space for visualization purposes in Figure 3.5.

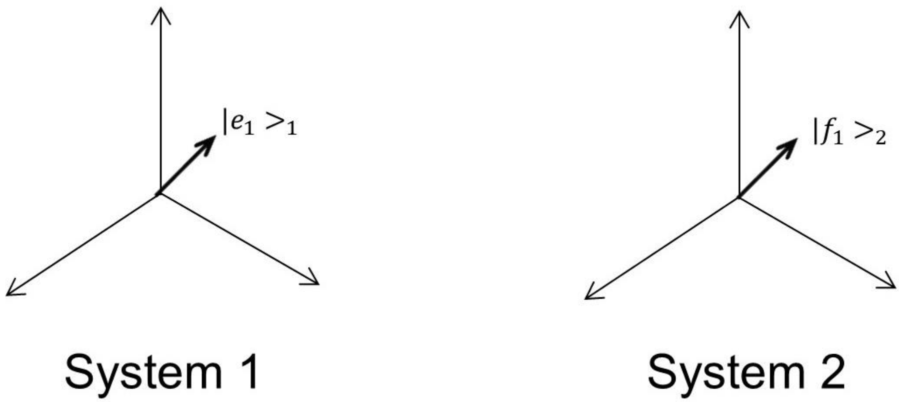

Figure 3.5: Geometric visualization of a product bipartite state with Schmidt vectors $|e_1\rangle_1$ and $|f_1\rangle_2$.

This can also be looked at as a single vector in the joint Hilbert space composing System 1-System 2, $\mathcal{H}_{1,2}$ as shown in Figure 3.6. In reality, the vector will exist in a higher than 3-dimensional space.

In an entangled system of particles in a pure state, the quantum state cannot be decomposed as a single product state. In Figure 3.7 an entangled bipartite system is illustrated that is composed of the superposition of two product states, i.e.,

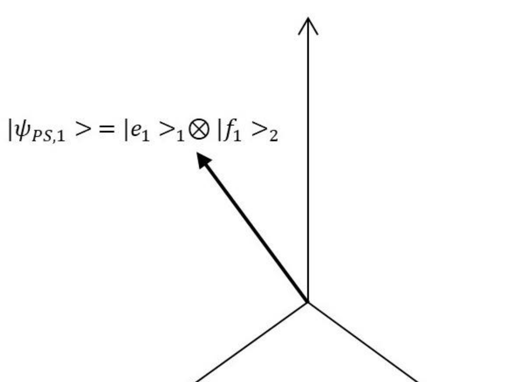

Figure 3.6: Vector representing a product state of two systems.

$$|\psi\rangle = \sqrt{a}|e_1\rangle_1 \otimes |f_1\rangle_2 + \sqrt{1-a}|e_2\rangle_1 \otimes |f_2\rangle_2 \text{ where } a \in \mathbb{C}, |a| \leq 1.$$

The dotted line between the vectors is shown to be weighted by the coefficients of each respective product state. There is a 90° angle between the orthogonal states $|e_1\rangle_1$ and $|e_2\rangle_1$ and similarly between $|f_1\rangle_2$ and $|f_2\rangle_2$. This entangled state can also be visualized as a superposition of $|\psi_{PS,1}\rangle = |e_1\rangle_1 \otimes |f_1\rangle_2$ and $|\psi_{PS,2}\rangle = |e_2\rangle_1 \otimes |f_2\rangle_2$, where

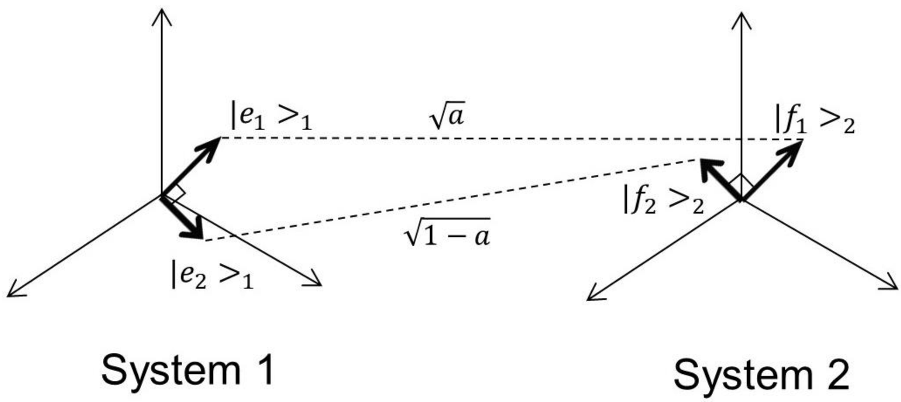

Figure 3.7: Geometric visualization of the Schmidt vectors composing an entangled bipartite system.

the subscript PS denotes product state. In this case, $|\psi\rangle = \sqrt{a}|\psi_{PS,1}\rangle + \sqrt{1-a}|\psi_{PS,2}\rangle$.

This is illustrated in Figure 3.8. The degree of the superposition $\sqrt{a}$ related to the entanglement, can be seen to simply correspond geometrically to the angle between the entangled state $|\psi\rangle$ and the product state $|\psi_{PS,1}\rangle$ via $\theta = \text{acos}(\sqrt{a})$. The three vectors $|\psi_{PS,1}\rangle$, $|\psi_{PS,2}\rangle$, and $|\psi\rangle$ can be seen to all lie in a single plane determined by any two of these vectors.

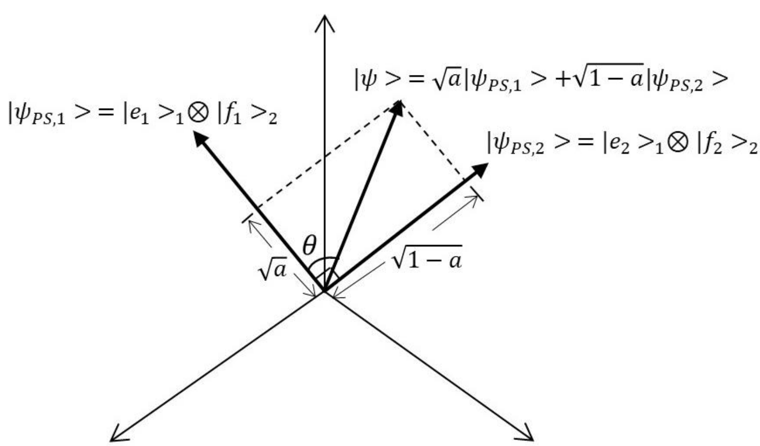

Figure 3.8: Geometry of an entanglement superposition state composed of two product states.