Partial Density Matrices

Consider a bipartite composite system in a tensor product state given by $\rho = \rho_1 \otimes \rho_2$. We denote the operation of partial trace $\mathrm{Tr}_j \rho$ whereby System j has been traced from ρ in a manner that $\mathrm{Tr}_1 \rho = \rho_2$, $\mathrm{Tr}_2 \rho = \rho_1$. Partial trace is a unique linear operator with these properties. It is useful to investigate how the density matrices of the various systems are affected by the 3-Step operations; herein the partial densities of Systems $1'$ and $2'$ are found after each of the three steps. Defining,

$$\begin{aligned}
\rho_{\mathrm{init},1'} &\equiv \mathrm{Tr}_{2'} \rho_{\mathrm{init}} \\
\rho_{\mathrm{init},2'} &\equiv \mathrm{Tr}_{1'} \rho_{\mathrm{init}} \\
\rho_{\mathrm{photon},B} &\equiv \mathrm{Tr}_{\mathrm{C}} |\psi_{\mathrm{photon,PBS}}\rangle\langle\psi_{\mathrm{photon,PBS}}| \\
\rho_{\mathrm{photon},C} &\equiv \mathrm{Tr}_{\mathrm{B}} |\psi_{\mathrm{photon,PBS}}\rangle\langle\psi_{\mathrm{photon,PBS}}|,
\end{aligned}$$

it can be shown (Exercise 3.17 in the book or kindle version of theQMP)

$$\rho_{\mathrm{init},1'} = \rho_{\mathrm{photon},B} \otimes \rho_{r,1}^{(0)} \tag{3.31}$$

and similarly,

$$\rho_{\mathrm{init},2'} = \rho_{\mathrm{photon},C} \otimes \rho_{r,2}^{(0)}.$$

Consider the density matrix representation of the states $|e_1^0\rangle_{1'}$ and $|e_2^0\rangle_{1'}$ given by

$$\begin{aligned}
\rho_{1,1'}^{(0)} &\equiv |1\rangle_{B\,B}\langle 1| \otimes \rho_{r,1}^{(0)} \\
\rho_{2,1'}^{(0)} &\equiv |0\rangle_{B\,B}\langle 0| \otimes \rho_{r,1}^{(0)}.
\end{aligned}$$

The Schrödinger's equation unitary evolution of ρ_{init} is given by

$$\rho_{\mathrm{fin},1} = (I \otimes W \otimes I \otimes V)\rho_{\mathrm{init}}\,(I \otimes W \otimes I \otimes V)',$$

where the identity matrices apply to the ancilla qubits. It follows that the density matrix representations of the evolves states $|e_1^1\rangle_{1'}$ and $|e_2^1\rangle_{1'}$ can be written

$$\begin{aligned}
\rho_{1,1'}^{(1)} &= W\rho_{1,1'}^{(0)}W' \\
\rho_{2,1'}^{(1)} &= W\rho_{2,1'}^{(0)}W'.
\end{aligned}$$

It can be shown (Exercise 3.18 in the book or kindle version of theQMP) after Step 1, the density matrix of System $1'$ denoted $\rho_{1'}^{(1)}$ is

$$\rho_{1'}^{(1)} = a\rho_{1,1'}^{(1)} + (1-a)\rho_{2,1'}^{(1)}. \tag{3.32}$$

Step 2 is applied, for which the resulting density matrix is given by

$$\rho_{\mathrm{fin},2} \equiv (X \otimes Y)\rho_{\mathrm{fin},1}(X \otimes Y)'.$$

Now, for the case of pure states, X maps both $\rho_{1,1'}^{(1)}$ and $\rho_{2,1'}^{(1)}$ to $\rho_{1,1'}^{(1)}$. Hence the density matrix of System $1'$ denoted $\rho_{1'}^{(2)}$ becomes

$$\rho_{1'}^{(2)} = \mathrm{Tr}_{A_1} X(|0\rangle_{A_1\,A_1}\langle 0| \otimes \rho_{1'}^{(1)})X'$$
$$\rho_{1'}^{(2)} = \rho_{1,1'}^{(1)}.$$

After Step 2, the density matrix of Qubit 1 is

$$\rho_{A_1}^{(2)} = \mathrm{Tr}_{1'} X(\rho_{A_1}^{(1)} \otimes \rho_{1'}^{(1)})X',$$

which is found to be

$$\rho_{A_1}^{(2)} = a\big|0\rangle_{A_1\,A_1}\langle 0\big| + (1-a)\big|1\rangle_{A_1\,A_1}\langle 1\big|.$$

Step 3 is the application of the observables representing Bell's experiment. Each of these observables can be written as a (different) tensor product $\mathcal{O} = \mathcal{O}_1 \otimes \mathcal{O}_2$ that is applied to the two ancilla qubits. Step 3 yields the expected value $\mathcal{E}(\rho_{\mathrm{init}}, \mathcal{O}) = \mathrm{Tr}\left((\mathcal{O}_1 \otimes \mathcal{O}_2)\rho_{A_1,A_2}\right)$, where ρ_{A_1,A_2} is the joint state of the composite system of the two ancillae after Step 2.