Specific Device-Particle Modeling

In Equation (3.1), a general class of Hamiltonians was considered for which a UMDT can be defined. Now, a specific Hamiltonian model of device-particle is considered both to illustrate how to apply these operations and also to understand how these operations address Hypothesis Tests 3.1 and 3.2.

Given that a single photon has impinged on the two devices as in Figure 3.4 assuming the Step 1 Schrödinger unitary evolution in Hypothesis H_0 is common in both Hypothesis Tests 3.1 and 3.2, the state will evolve to Equation (3.9). Step 2 is to apply the local unitary X to Device 1 and Y to Device 2 that swaps the unitarily predicted entanglement between System $1'$ and System $2'$, into entanglement between the two ancilla qubits A_1 and A_2. Step 3 is to conduct a Bell measurement on these two qubits.

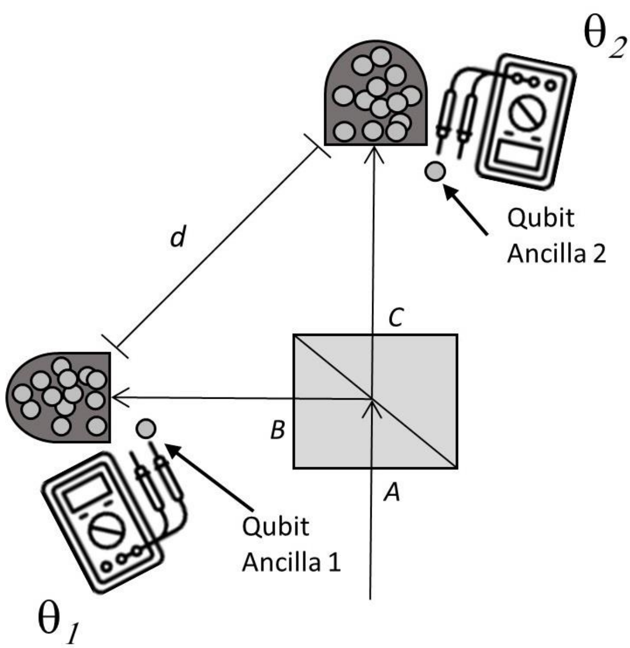

Figure 3.15: Step 3: Bell measurement on Ancilla Qubits.

Consider first a model for which each device is composed of two spin ½ particles with states that lie in a 4-dimensional Hilbert space. The relevant electromagnetic field in Figure 3.4 can be specified by the B and C photon modes and the two polarization modes for which the states lie in a 4-dimensional Hilbert space.

The Hamiltonian then is given by,

$$H = H_F \otimes I_4 \otimes I_4 + I_4 \otimes H_1 \otimes I_4 + I_4 \otimes I_4 \otimes H_2 + H_{int} \qquad (3.14)$$

where I_j is the j by j identity matrix. Denote the spin ½ operators in terms of the Pauli matrices and raising and lowering operators as

$$S_x = \frac{1}{2}\sigma_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, S_y = \frac{1}{2}\sigma_y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, S_z = \frac{1}{2}\sigma_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$S_+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, S_- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

We consider a two-spin Heisenberg model for the Hamiltonian of each device

$$H_i = \Omega_{1,i} S_z \otimes I_2 + \Omega_{2,i} I_2 \otimes \sigma_z + J_{x,i} S_x \otimes S_x + J_{y,i} S_y \otimes S_y + J_{z,i} S_z \otimes S_z. \tag{3.15}$$

Note that interactions between the particles composing each device have been included via $\{J_{x,i}, J_{y,i}, J_{z,i}\}$. The relevant electromagnetic field free Hamiltonian is given by

$$H_F = w a_B{}^\dagger a_B \otimes I_2 + w I_2 \otimes a_C{}^\dagger a_C,$$

where $a_B{}^\dagger$ (a_B) is an operator that creates (annihilates) a photon of vertical polarization in the *B* Port of the beam splitter shown in Figure 3.4 and $a_C{}^\dagger(a_C)$ creates (annihilates) a photon of horizontal polarization in the *C* Port of the beam splitter. The cases of a horizontally polarized photon in the *B* output port or vertical polarized photon in the *C* Port can be included if desired; however, as the input to the beam splitter in the *Z* Port of Figure 3.1 is assumed to be the vacuum state, such states cannot occur via Schrödinger's equation.

Consider the following field-particle interaction model,

$$H_{F,1} = c_1(a_B \otimes I_2 \otimes S_+ \otimes I_2 \otimes I_4 + a_B{}^\dagger \otimes I_2 \otimes S_- \otimes I_2 \otimes I_4)$$
$$H_{F,2} = c_2(I_2 \otimes a_c \otimes I_4 \otimes S_+ \otimes I_2 + I_2 \otimes a_C{}^\dagger \otimes I_4 \otimes S_- \otimes I_2)$$

for which $H_{int} = H_{F,1} + H_{F,2}$. These interaction terms have the effect of interacting the photon mode that impinges on a device with the first of the two spin particles that compose the device. However, as the two spins that compose that device can themselves interact via the coefficients $\{J_{x,i}, J_{y,i}, J_{z,i}\}$ in Equation (3.15) a photon that impinges on a device can affect both particles.

Let the initial state of the ancillae, EM field, Device 1, Device 2 be

$$|\psi_{\text{init}}\rangle = \sqrt{a}|0\rangle_{A_1} \otimes |1_V\rangle_B \otimes |\psi_r^0\rangle_1 \otimes |0\rangle_{A_2} \otimes |0_H\rangle_C \otimes |\psi_r^0\rangle_2 + \sqrt{1-a}|0\rangle_{A_1} \otimes |0_H\rangle_B \otimes |\psi_r^0\rangle_1 \otimes |0\rangle_{A_2} \otimes |1_H\rangle_C \otimes |\psi_r^0\rangle_2 \tag{3.16}$$

where $|\psi_r^0\rangle_i \in \mathcal{H}_i$, *i=1,2*, $\mathcal{D}(\mathcal{H}_i) = 2$,

$$|0\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |1\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Consider the initial device states of $|\psi_r^0\rangle_1 = |0\rangle \otimes |0\rangle$, $|\psi_r^0\rangle_2 = |0\rangle \otimes |1\rangle$. The Schrödinger unitary evolution operator to be applied to $|\psi_{\text{init}}\rangle$ is given by

$$U(t) = e^{-iHt} = (I \otimes W) \otimes (I \otimes V).$$

For a fixed Schrödinger unitary evolution time $t = \tau_e$, the initial state is evolved in Step 1 to $|\psi_{\text{fin},1}\rangle = U(\tau_e)|\psi_{\text{init}}\rangle$ or

$$|\psi_{\text{fin},1}\rangle = \sqrt{a}\,|0\rangle_{A_1} \otimes |e_1^1\rangle_{1'} \otimes |0\rangle_{A_2} \otimes |f_1^1\rangle_{2'} + \sqrt{1-a}|0\rangle_{A_1} \otimes |e_2^1\rangle_{1'} \otimes |0\rangle_{A_2} \otimes |f_2^1\rangle_{2'}.$$

The operators $T_1$and T_2 of Step 2 are used to transfer the entanglement between Systems $1'$ and $2'$ to entanglement between the two ancilla qubits:

$$T_1 = (|1\rangle_{A_1} \otimes |e_1^1\rangle_{1'})(\langle 0|_{A_1} \otimes {}_{1'}\langle e_1^1|) + (|0\rangle_{A_1} \otimes |e_1^1\rangle_{1'})(\langle 0|_{A_1} \otimes {}_{1'}\langle e_2^1|)$$
$$T_2 = (|1\rangle_{A_2} \otimes |f_2^1\rangle_{2'})(\langle 0|_{A2} \otimes_{2'} \langle f_2^1|) - (|0\rangle_{A_2} \otimes |f_2^1\rangle_{2'})(\langle 0|_{A_2} \otimes {}_{2'}\langle f_1^1|).$$

The vectors $|e_i^1\rangle_{1'}, |f_i^1\rangle_{1'}, i = 1{,}2$ can be computed from Equation (3.8) and for this model are:

$$|e_1^1\rangle_{1'} = W|1_V\rangle_B \otimes |0\rangle \otimes |0\rangle,\ |e_2^1\rangle_{1'} = W|0_V\rangle_B \otimes |0\rangle \otimes |0\rangle$$
$$|f_1^1\rangle_{2'} = V\,|0_H\rangle_C \otimes |0\rangle \otimes |0\rangle,\ |f_2^1\rangle_{2'} = V\,|1_H\rangle_C \otimes |0\rangle \otimes |0\rangle.$$

Upon extending T_1 and T_2 to unitary matrices via the procedure previously shown, i.e., adding orthogonal rows to complete the basis, we arrive at X and Y. Applying the unitary $X \otimes Y$ to $|\psi_{\text{fin}}\rangle$ will transfer the state in the two ancilla qubits to become entangled at the expense of the loss of entanglement between Systems $1'$ and $2'$.

In Step 3, a Bell experiment is performed on the two ancilla qubits. Steps 1 and 2 were simulated for $t = .4$ using the Hamiltonian parameters $w = 2$, $\{\Omega_{1,1,}\Omega_{2,1,,}\mathrm{J}_{x,1}, \mathrm{J}_{y,1}, \mathrm{J}_{z,1}\} = \{.5, .2, .1, .3, .5\}$ and $\{\Omega_{1,2,}\Omega_{2,2,}J_{x,2}, J_{y,2}, J_{z,2}\} = \{.3, .4, .7, .2, .6\}$. Two initial device states were chosen as $|\psi_r^0\rangle_1 = |0\rangle \otimes |0\rangle$, $|\psi_r^0\rangle_2 = |0\rangle \otimes |1\rangle$. Results of the Bell experiment are given in terms of the CHSH sum in Figure 3.16 for both the Step 1 Schrödinger unitary prediction for *t*=.4 and under an assumption that the photon takes a definite path via either the *B* or *C* Port. One sees that the CHSH sum is always greater than the case of a known photon path, for any initial superposition of the photon. Furthermore, the CHSH sum is the same independent of the initial state, the time of the unitary evolution, the number of particles composing the devices, the interactions within the device, the interactions between the photon and particles composing the device. In short, the argument given is rigorous.

Since the entanglement has been transferred to the two ancilla qubits, the unitary prediction can also be calculated analytically using a two-qubit version of the CHSH inequality [102]

$$S = \left|\hat{C}(a,b) + \hat{C}(a,\acute{b}) + \hat{C}(\acute{a},b) - \hat{C}(\acute{a},\acute{b})\right| \leq 2, \tag{3.17}$$

where $a, \acute{a}$ and $b, \acute{b}$ are the two-valued variables (± 1) for the first and second qubits respectively. Quantum mechanically, the correlation between a and b is given in terms of the system density matrix ρ and the Hermitian operators $\hat{a}$ and $\hat{b}$ corresponding to a and b:

$$\hat{C}(a,b) = \mathrm{Tr}\big(\rho(\hat{a}\otimes\hat{b})\big). \tag{3.18}$$

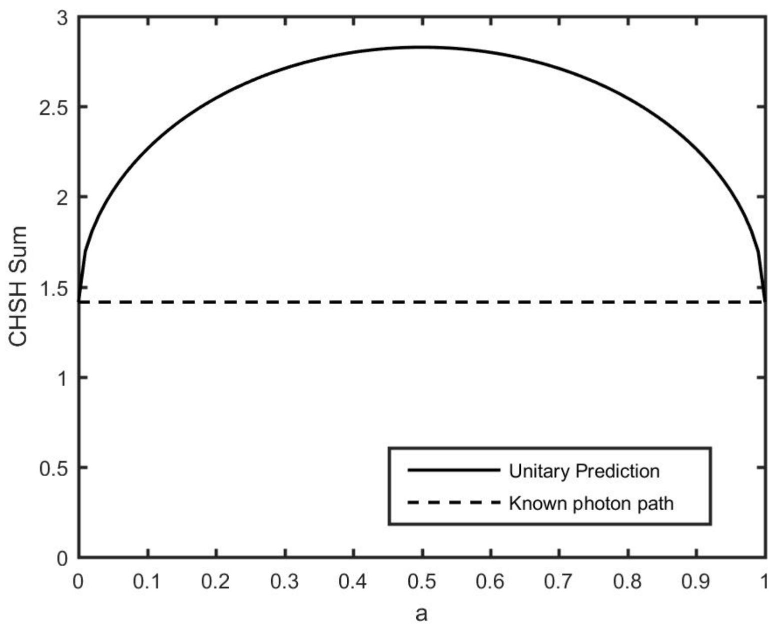

Figure 3.16: CHSH Sum versus degree of photon superposition, Unitary CHSH sum is $2\sqrt{2}$ $at\ a = .5$ whereas for known photon path of either $|\psi^1\rangle_1 \otimes |\psi^0\rangle_2$ or $|\psi^0\rangle_1 \otimes |\psi^1\rangle_2$ the CHSH sum is $\sqrt{2}$ as seen in the dotted curve, independent of a.

The two-qubit measurement settings corresponding to the polarization measurement settings of Equation (3.3) can be shown to be given by

$$a = \sigma_z, \qquad \acute{a} = \sigma_x \tag{3.19}$$

$$b = \frac{\sigma_z + \sigma_x}{\sqrt{2}}, \ \acute{b} = \frac{\sigma_z - \sigma_x}{\sqrt{2}}. \tag{3.20}$$

Using the entangled pure state

$$|\psi\rangle = \sqrt{a}|00\rangle + \sqrt{1-a}|11\rangle, \tag{3.21}$$

the CHSH sum for the two ancillary qubits is found from the above equations to be

$$S = \sqrt{2}(1 + 2\sqrt{a(1-a)}). \tag{3.22}$$

As expected, Equation (3.22) is the same as the simulation of the unitary prediction in Figure 3.16. The entanglement between two qubits can be quantified in terms of Wootters' *concurrence* [103] which, for the pure state of Equation (3.21) is given by

$$\mathbb{C} = 2\sqrt{a(1-a)} \tag{3.23}$$

so that the CHSH sum can be rewritten as

$$S = \sqrt{2}(1 + \mathbb{C}). \tag{3.24}$$

The CHSH sum for the known photon path is

$$S_{Cl} = \sqrt{2}. \tag{3.25}$$

Therefore, at this measurement setting, the fractional difference between the unitary and known photon path CHSH sums is exactly given by the entanglement of the state in the form of the concurrence

$$\frac{S_U - S_{Cl}}{S_{Cl}} = \mathbb{C}. \tag{3.26}$$

The CHSH sum in Equation (3.24) also results using a mixed state ρ with the same concurrence $\mathbb{C}(\rho) = 2\sqrt{a(1-a)}$. This can be seen from the result [103] that a mixed state of two qubits can be decomposed into a convex sum of pure states, all with concurrence equal to the concurrence of the mixed state

$$\rho = \sum_i c_i |\psi_i\rangle\langle\psi_i| \tag{3.27}$$

with $\sum_i c_i = 1.$ Since each $|\psi_i\rangle$ is unitarily equivalent to a Schmidt decomposition of the form of Equation (3.21), the properties of the trace in Equation (3.18) imply that the mixed state CHSH sum is identical to that of the pure state result in Equation (3.24) at this measurement setting

$$S_\rho = \sqrt{2}\big(1 + \mathbb{C}(\rho)\big) = \sqrt{2}\left(1 + 2\sqrt{a(1-a)}\right). \tag{3.28}$$