Consistent Histories

Consistent histories is a theory that provides a means of generating potential histories of the evolution of a closed system in a Hilbert space $\mathcal{H}$. The closed system may include not only a system that is being measured, but also the device. We will describe a version of consistent histories for which the initial state at time t_0 is a pure state given by $|\psi_0\rangle$ and evolves from time t_A to time t_B via the unitary operator $U(t_A, t_B)$, except at a set of times denoted $\{t_1, t_2, \cdots, t_N\}$, for which the state undergoes projection, and N is the number of projections.

There are often more than one set of potential histories. Potential histories will be indexed by $k = 1,2,\cdots$. One can define a sample space $\mathbb{S}_k$ that contains elements, each of which is a set of projections that might occur at the times $\{t_1, t_2, \cdots, t_N\}$. A quantum sample space $\mathbb{S}_k$ consists of a set of histories that are consistent in a manner that a probability can be assigned to each history such that the probabilities of each element or individual history, sum to unity. One can see that if the elements of $\mathbb{S}_k$are disjoint or mutually exclusive and also if the union of the elements is the entire sample space, then one is guaranteed that the sum of the element probabilities in $\mathbb{S}_k$ will be unity. Sufficient conditions under which the elements of $\mathbb{S}_k$ are mutually exclusive will be found.

Suppose the initial state is given by $|\psi_0\rangle$ and that for time t_i for histories belonging to $\mathbb{S}_k$, the set of possible projections are denoted by $P_1(k, t_i), P_2(k, t_i), \cdots$. A set of projectors called histories $\{Y_{\alpha_1}(\mathrm{k}), Y_{\alpha_2}(\mathrm{k}), \cdots\}$ are defined at various times as

$$Y_{\alpha_j}(k) = P_0 \otimes P_{\alpha_{1,j}}(k, t_1) \otimes P_{\alpha_{2,j}}(k, t_2) \otimes \cdots \otimes P_{\alpha_{N,j}}(k, t_N)$$

where $\alpha_j = \left(\alpha_{1,j}, \alpha_{2,j}, \cdots, \alpha_{N,j}\right)$, and $P_0 \equiv |\psi_0\rangle\langle\psi_0|$. Now we can define the quantum sample history space $\mathbb{S}_k = \left\{Y_{\alpha_1}(k), Y_{\alpha_2}(k), \cdots, Y_{\alpha_N}(k)\right\}$ as consisting of the set of events that form a Boolean algebra with the properties

$$\sum_i Y_{\alpha_i}(k) = I$$
$$Y_{\alpha_i}(k) Y_{\alpha_j}(k) = 0, \;\; \forall i \neq j. \tag{4.14}$$

The events are assumed [134, p. 57] to be mutually orthogonal via Equation (4.14). As the events are mutually orthogonal, they also commute and the events of $\mathbb{S}_k$ are a Boolean algebra for which the conjunction and disjunction operations $\wedge$ and $\vee$ respectively are well-defined.

Events defined at times $\{t_1, t_2, \cdots, t_N\}$ in a given experiment should be consistent in the sense that a probability can be assigned to any particular history and the histories within $\mathbb{S}_k$ are disjoint. This later condition is not the same as the mutual orthogonal condition of Equation (4.14). A chain operator $K(Y_{\alpha_j}(k))$ is defined as

$$K\left(Y_{\alpha_j}(k)\right) = P_{\alpha_{N,j}}(k, t_N) U(t_N, t_{N-1}) P_{\alpha_{N-1,j}} \cdots P_{\alpha_{2,j}}(k, t_2) P_{\alpha_{1,j}}(k, t_1) U(t_1, t_0) P_0.$$

The probability that the history $Y_{\alpha_j}(k)$ occurs is given by

$$p(Y_{\alpha_j}(k)) = \mathrm{Tr}\left[K\left(Y_{\alpha_j}(k)\right)^{\dagger} K\left(Y_{\alpha_j}(k)\right)\right].$$

A sufficient condition [135] [136] that a given sample space $\mathbb{S}_k$ represents a set of consistent histories (that are mutually exclusive and for which the probabilities of the elements sum to unity) is that the different histories are mutually exclusive in the sense that

$$\mathrm{Tr}\left[K\left(Y_{\alpha_j}(k)\right)^{\dagger} K\left(Y_{\alpha_l}(k)\right)\right] = 0, \ \ \forall\, j \neq l. \tag{4.15}$$

Note that this condition restricts within each k the set of histories (see the related Exercises 4.6-4.8 in the book or kindle version of theQMP). When the individual histories within a given set of consistent histories satisfy Equation (4.15) the probability of any two histories will be additive via

$$p\left(Y_{\alpha_j}(k) \vee Y_{\alpha_l}(k)\right) = p\left(Y_{\alpha_j}(k)\right) + p\left(Y_{\alpha_l}(k)\right), \qquad \forall\, j \neq l,$$

or

$$p\left(Y_{\alpha_j}(k) \wedge Y_{\alpha_l}(k)\right) = 0 \ \ \forall\, j \neq l.$$

A history may or may not be Schrödinger unitary. For a given problem with a Hamiltonian H if there was only a valid consistent history that was Schrödinger unitary, then the measurement problem would be solved by consistent history theory. However, as has been raised by Dowker and Kent [137], this is not the case in light of the non-unique consistent history sets that can be defined when $k > 2$:

See the print edition of The Quantum Measurement Problem for quotation.

Griffiths considers this criticism by Dowker and Kent in [138]. He states

> *Dowker and Kent conclude that the consistent history approach to quantum theory lacks predictive power, and for this reason is not satisfactory as a fundamental scientific theory. ... it is the fact that the consistent history approach, as a fundamental theory, treats all frameworks or consistent families "democratically", and provides no criterion to select out one in particular. In other words, there is no "law of nature" which specifies the framework.*

Griffiths concedes that

... consistent histories, as a fundamental theory of nature, does not single out a particular framework.

Griffiths gives an example of consistent histories regarding Dowker and Kent's criticism which is similar to what we have already been examining in the UMDT of Chapter 3. That is, a photon is considered that enters a beam splitter at A at time t_0 in Figure 3.4 and exits at time t_1 in one of two output paths B and C. In each path B and C there is a detector. The initial joint state of the photon plus Device 1 plus Device 2 is given by $|\Psi_0\rangle$ and the initial ready state of Device i is assumed to be a pure state and is denoted by $|\psi_r^0\rangle_i$, *i=1,2*, $|\psi_r^0\rangle_i \in \mathcal{H}_i$, *i=1,2*, $\mathcal{D}(\mathcal{H}_i) = N$. For simplicity, it will be assumed that the detectors are 100% absorbing, that is the device always completely absorbs the photon. Suppose that in both paths there is only the vacuum state $|0\rangle$. At time t_1 it is assumed that the initial ready device state $|\psi_r^0\rangle_i$ changes to a read-out state that is indicative of no measurement and is denoted $|\psi^0(t_1)\rangle_i, i = 1{,}2$, for which it changes to $|\psi^0(t_2)\rangle_i, i = 1{,}2$. Consider the case that a photon with state $|1_V\rangle_B$ is inserted in Path B. The state of Device 1 evolves to its readout state indicating that the device has detected a photon, which will be denoted $|\psi^1\rangle_1 \in \mathcal{H}_1$. Similarly, if a photon with state $|1_H\rangle_C$ is inserted solely in Path C and Device 2 is 100% absorbing, the final state of Device 2 is denoted as $|\psi^1(t_2)\rangle_2 \in \mathcal{H}_2$.

Consider the initial local state $(|1_V\rangle_A + |1_H\rangle_A)/\sqrt{2}$ of the photon in mode A at time t_0. This state then propagates through the beam splitter and evolves to the state $|\psi_{photon,PBS}\rangle = (|1_V\rangle_B \otimes |0\rangle_C + |0\rangle_B \otimes |1_H\rangle_C)/\sqrt{2}$ at time t_1 which represents an equal superposition of output paths B and C. Let $|\Psi_0\rangle = |a\rangle \otimes |\psi_r^0\rangle_1 \otimes |\psi_r^0\rangle_2$. Denote the projection operator of an arbitrary state $|\psi\rangle$ by $\mathcal{P}(|\psi\rangle) \equiv |\psi\rangle\langle\psi|$. We define $|\psi_{fin,1}\rangle$ consistent with previous notation as:

$$|\psi_{fin,1}\rangle = |0\rangle \otimes (|\psi^1(t_2)\rangle_1 \otimes |\psi^0(t_2)\rangle_2 + |\psi^0(t_2)\rangle_1 \otimes |\psi^1(t_2)\rangle_2)/\sqrt{2}.$$

Griffiths shows two sets of histories that are individually consistent. One set (denoted by $k = 1$) can be found via the following four histories:

$$Y_{\alpha_1}(1) = \left\{\mathcal{P}(|\Psi_0\rangle), \mathcal{P}\left(|\psi_{photon,PBS}\rangle \otimes |\psi^1(t_1)\rangle_1 \otimes |\psi^1(t_1)\rangle_2\right), \mathcal{P}(|\psi_{fin,1}\rangle)\right\}$$

$$Y_{\alpha_2}(1) = \left\{\mathcal{P}(|\Psi_0\rangle), \mathcal{P}\left(|\psi_{photon,PBS}\rangle \otimes |\psi^0(t_1)\rangle_1 \otimes |\psi^0(t_1)\rangle_2\right), I - \mathcal{P}(|\psi_{fin,1}\rangle)\right\}$$

$$Y_{\alpha_3}(1) = \left\{\mathcal{P}(|\Psi_0\rangle), I - \mathcal{P}\left(|\psi_{photon,PBS}\rangle \otimes |\psi^0(t_1)\rangle_1 \otimes |\psi^0(t_1)\rangle_2\right), \mathcal{P}\left(|\psi_{fin,1}\rangle\right)\right\}$$

$$Y_{\alpha_4}(1) = \left\{\mathcal{P}(|\Psi_0\rangle), I - \mathcal{P}\left(|\psi_{photon,PBS}\rangle \otimes |\psi^0(t_1)\rangle_1 \otimes |\psi^0(t_1)\rangle_2\right), I - \mathcal{P}(|\psi_{fin,1}\rangle)\right\}.$$

However,

$$p\left(Y_{\alpha_2}(1)\right) = p\left(Y_{\alpha_3}(1)\right) = p\left(Y_{\alpha_4}(1)\right) = 0$$

where $p(Y_\alpha)$ denotes the probability of history Y_α occurring. Hence for $k = 1$ there is only one non-zero probability history given by $Y_{\alpha_1}(1)$. For the set $k = 2$ the following two sets of non-zero probability histories (along with a correspond set of zero probability histories) are found:

$$Y_{\alpha_1}(2) = \begin{Bmatrix} \mathcal{P}(|\Psi_0\rangle), \mathcal{P}\left(|\psi_{photon,PBS}\rangle \otimes \left|\psi^0(t_1)\right\rangle_1 \otimes \left|\psi^0(t_1)\right\rangle_2\right), \\ \mathcal{P}\left(|0\rangle \otimes |\psi^1(t_2)\rangle_1 \otimes |\psi^0(t_2)\rangle_2\right) \end{Bmatrix}$$

$$Y_{\alpha_2}(2) = \begin{Bmatrix} \mathcal{P}(|\Psi_0\rangle), \mathcal{P}\left(|\psi_{photon,PBS}\rangle \otimes \left|\psi^0(t_1)\right\rangle_1 \otimes \left|\psi^0(t_1)\right\rangle_2\right), \\ \mathcal{P}\left(|0\rangle \otimes |\psi^0(t_2)\rangle_1 \otimes |\psi^1(t_2)\rangle_2\right) \end{Bmatrix}.$$

Note that the set of histories for $k = 2$ are mutually exclusive in the sense that

$$p\left(Y_{\alpha_j}(2) \cup Y_{\alpha_l}(2)\right) = p\left(Y_{\alpha_j}(2)\right) + p\left(Y_{\alpha_l}(2)\right), \qquad \forall\, j \neq l.$$

However, it can be the case that

$$p\left(Y_{\alpha_j}(1) \cup Y_{\alpha_l}(2)\right) \neq p\left(Y_{\alpha_j}(1)\right) + p\left(Y_{\alpha_l}(2)\right).$$

In this sense, the two sets of histories $k = 1$ and $k = 2$ are termed incompatible. A sufficient condition for the incompatibility is that the projectors at a given time do not commute when chosen from two different sets of histories.

Griffiths states in [138]:

> *There are many (in fact, an uncountably infinite number of) other frameworks which could have been employed to discuss the same situation. The fact that the same physical system can be discussed using many different frameworks gives rise to the problem of choice: how does one decide which framework is appropriate for describing what goes on in this closed quantum system?*

The discrimination of the incompatible sets of histories is required and incumbent upon the theory to provide. A Category 1 theory would provide such discrimination in terms of meeting Requirement R1.3 (or Requirement R2.2 for a Category 2 theory). In terms of using actual data from measurements to determine the physical history, Griffiths states [138]:

> *Checking the predictions given by different incompatible frameworks always involves alternative experimental arrangements… In any case, just as there is no single "correct" choice of consistent family for describing the system, there is no single arrangement of*

apparatus which can be used to verify the predictions obtained using different families.

We disagree with this statement. The UMDT of Chapter 3 is a single experimental arrangement that one can use to determine via the CHSH sum whether or not the history came from either the set corresponding to $k = 1$ or the set corresponding to $k = 2$.

Note that the use of multiple potential histories would not necessarily be a major problem if one addresses the philosopher's measurement problem, which is an *a posteriori* problem given that a measurement has occurred in a given basis. This is because if one is given the measurement basis and given the measurement times, then one can determine the histories that are consistent with any given experiment and rule out incompatible histories. This would render the measurement problem an interpretational problem as opposed to the physical measurement problem that we have defined. Regarding this issue, Griffiths states [138]

The endless discussions about how to interpret the state $|S\rangle$ and resolve the corresponding "measurement" problem have been rightly criticized by Bell.

Presumably, the "endless discussions" that Griffiths refers to include Einstein's objections, Schrödinger's cat paper, and the substantial body of work that concentrates on the issue of the lack of a product state prediction under unitary evolution. The physical measurement problem would appear to be irrelevant to Griffiths, as he is of the viewpoint that current quantum mechanics is correct FAPP. This viewpoint leads directly to the rejection of the physical measurement problem in favor of the philosophers' definition of the measurement problem.

On the other hand, consistent histories is a rather interesting development that does appear to go beyond the Copenhagen interpretation in providing sets of consistent histories. As noted by Dowker and Kent [137] consistent histories may be useful:

The virtues of the consistent histories approach are worth reasserting. We have a natural mathematical criterion which is empirically supported and which identifies the physical content of quantum theory to be propositions about particular collections of events encoded in the consistent sets. This gives a framework in which attempts to set out a quantum theory of the universe, and the problems inherent in this idea, can be sensibly discussed…. Many attempts have been made to find natural mathematical structures and interpretational postulates that allow one to use quantum theory to make statements about physics that go beyond the Copenhagen bounds. In our view, the consistent histories formalism is one of the most significant developments.

Consistent histories as developed by Griffiths does indeed narrow down the set of possible histories as histories via requiring Equation (4.15) to be fulfilled. Hence Category 1 or 2 theories that violate this rule must be excluded. Consistent histories is certainly a step in the right direction regarding the physical measurement problem as we agree that it narrows down the potential sets of histories that need to be considered, in the sense that a theory that resolves the measurement problem must produce a history that is within one of the sets of consistent histories. However, there are a large number of potential histories that remain after imposing Equation (4.15) and without any additional assumptions, consistent histories is simply silent on further narrowing down to a single consistent set.