## *Ghirardi, Rimini, and Weber (GRW)*

There are a number of theories that address the measurement problem through spatial localization. Transposing the measurement problem completely to the problem of spatially induced measurement localization is relevant if spatial localization is a property that is related to the resolution of the entanglement predicted under unitary evolution versus the product state that occurs under measurement. There is an implicit assumption being made by purveyors of such theories, that this is the case.

Localization may or may not be a necessary condition for positive measurement. However, as macroscopic objects are generally seen as localized in classical physics, it seems reasonable to examine mechanisms and theories that would localize a wave function when positive measurement occurs.

With this in mind, consider theories that address the localization of objects. A theory proposed in 1986 by Ghirardi, Rimini, and Weber (GRW) put forward a model for spontaneous localization theories that utilize a stochastic methodology [224]. Such theories are a valid attempt to solve the measurement problem and are not simply a unitary interpretation that also produces Born's rule *a posteriori* given that a measurement has occurred. The original mathematical description in the paper [224] has since been considerably mathematically simplified. Bell [225] indicated that the original GRW proposal could also be viewed from the effect of the wave function as being transformed to another wave function by the multiplication of a Gaussian function. This was subsequently incorporated into the paper [226]. Consider $N$ particles with wave function

$$\psi\left(x^{(1)}, x^{(2)}, \cdots, x^{(N)}, t\right). \qquad (4.23)$$

The position of the $i$ th particle is given by $x^{(i)}$. Over a given time interval $\tau$, the wave function either evolves according to Schrödinger's equation or undergoes a stochastic process referred to as a hitting process in [224]. For a single particle, the hitting process is described by a Poisson process with average number of events per time interval $\tau$ described by the parameter $\lambda_h$. For $N$ particles the average number of events per time interval $\tau$ is $N\lambda_h$. Suppose a hit occurs to the $i$th particle at time $t = t_h$. Let $\hat{\psi}(x^{(1)}, x^{(2)}, \cdots, x^{(i)}, \cdots, x^{(N)}, t_h)$ be the wave function had there not been a hit, i.e. the wave function that is found from Schrödinger's equation. The function $\phi$ is defined by

$$\phi\left(x^{(1)}, x^{(2)}, \cdots, x^{(i)}, \cdots, x^{(N)}, t_h, z\right) \equiv h(x^{(i)}, z)\, \hat{\psi}\left(x^{(1)}, x^{(2)}, \cdots, x^{(i)}, \cdots, x^{(N)}, t_h\right)$$

whereby $h(x^{(i)}, z)$ is a positive function parameterized by $z$ that multiplies the wave function. It is assumed that $z$ is a real random variable with probability distribution function given by

$$p(z) = \frac{r(z)}{\int r(z)dz}$$

where

$$r(z) \equiv \int \cdots \int |\phi(x^{(1)}, x^{(2)}, \cdots, x^{(i)}, \cdots, x^{(N)}, t_h, z)|^2 dx^{(1)} \cdots dx^{(N)}.$$

Let the random variable $z$ be chosen stochastically with result $z = z_h$. The new wave function at time $t_h$ is given by

$$\psi\left(x^{(1)}, x^{(2)}, \cdots, x^{(i)}, \cdots, x^{(N)}, t_h\right) = \frac{\phi\left(x^{(1)}, x^{(2)}, \cdots, x^{(i)}, \cdots, x^{(N)}, t_h, z_h\right)}{\sqrt{r(z_h)}}.$$

In the case of GRW, it is assumed that $h(x^{(i)}, z)$ is a Gaussian probability function:

$$h\left(x^{(i)}, z\right) = f\left(x^{(i)} \middle| z\right) = \frac{1}{\sqrt{2\pi\sigma_h{}^2}} e^{-\frac{1}{2\sigma_h{}^2}\left(x^{(i)} - z\right)^2}.$$

For simplicity, it will be assumed that

$$h\left(x^{(i)}, z\right) = \sqrt{f\left(x^{(i)} \middle| z\right)},$$

for which $\int r(z)dz$ reduces to unity. Note that $p(z)$ will generally be peaked with $\psi$ which is related to where the particle would be found in position space under Schrödinger's equation. Hence by choosing a particular $z = z_h$, GRW theory will aid in localizing the wave function and decrease the effect of the superposition terms predicted under unitary evolution. Also note that as the number of events per time interval $\tau$ is $N\lambda$, the probability of a hit will increase with the number of particles that forms a device and for which all are predicted within Schrödinger's equation to go into superposition. For example, consider a macroscopic device that has a physical pointer in position space that changes its pointer from an initial Position 1 to a final Position 2 upon interacting with a particle. If the pointer is on the order of $N \sim 10^{23}$ particles, then all these particles will either be found in Position 1 or Position 2 in the conventional theory. Hence GRW theory for large $N$ rapidly helps in localizing, with the exception of the tails of the Gaussian distribution, which form a positional superposition.

*GRW as a POVM*

It was shown in [227] that there exists a POVM representation for a range of GRW-like dynamical reduction models. The proof is based on the Riesz representation theorem. Here we express GRW as a POVM based on the development in [228] from which can be derived several features of GRW. A single particle is assumed for simplicity; however, the theory can be extended to multiple particles.

We introduce here the use of continuous kets $|\Psi(x, t)\rangle$ which are a function of the wave functions of Equation (4.23). That is, we define the ket

$$|\Psi(x,t)\rangle \equiv \int \psi(x,t_h)|x\rangle dx$$

where $\psi(x,t_h)$ is a complex wave function of the form of Equation (4.23) with $x \in (-\infty,\infty)$, $\int |\psi(x,t_h)|^2 dx$=1. The kets $|x\rangle$ represent continuous position eigen-kets and satisfy $\langle x|x'\rangle = \delta(x-x')$ where $\delta(x-x')$ is the Dirac delta function. Similar to above, $|\widehat{\Psi}(x,t)\rangle$ denotes the continuous ket assuming no hits have occurred at time $t_h$.

Consider the measurement of $|\Psi(x,t)\rangle$ by positive operators of the form

$$M_i = \int_{-\infty}^{\infty} \sqrt{f(x|z_i)}\,|x\rangle\langle x|dx$$

and each $M_i$ $(i = 1, \ldots, N_p)$ is associated with a value of $z_i$ which offsets the center of the Gaussian. Note that

$$\int M_i{}' M_i dz = I.$$

The *i* th experimental outcome associated with the POVM at time $t = t_h$ is the projection of the state onto $M_i$, i.e.

$$|\Psi(x,t_h)\rangle = \frac{M_i|\widehat{\Psi}(x,t_h)\rangle}{||M_i|\widehat{\Psi}(x,t_h)\rangle||}. \tag{4.24}$$

$$|\Psi(x,t_h)\rangle = \frac{\int_{-\infty}^{\infty} \phi(x,t_h,z_i)\,|x\rangle\langle x|dx}{\sqrt{r(z_i)}}.$$

The probability of the *i* th outcome, denoted $p_i$, is given by

$$\begin{aligned} p_i &= \mathrm{Tr}[M_i^{\prime M_i}|\widehat{\Psi}(x,t_h)\rangle\langle\widehat{\Psi}(x,t_h)|] \\ &= \int_{-\infty}^{\infty} |\phi(x,t_h,z_i)\,|^2 dx \\ &= r(z_i). \end{aligned}$$

We see that the POVM formulation is identical to the GRW formulation. Note that the *i* th outcome of the POVM defines the position of the Gaussian $z_i$. In terms of what outcome occurs by a measurement in the POVM formulation of GRW, it is the particular position $z_i$ that is chosen from the Gaussian that is used to project the state.

The projections of the state are accomplished via the operators $M_i$ that are diagonal in the position basis. To further understand the effect of GRW, consider simplifying from the positional infinite dimensional Hilbert space consisting of the real numbers, to an approximation over a finite interval $x \in [-A, A]$ whereby the positions occur on a finite partition $(x_1 < x_2 < \cdots < x_N)$ that is equally spaced $\left|x_{j+1} - x_j\right| = \Delta$, $j = 1, \cdots N-1$. One can convert the integral representations of the operators to summations and the operators $M_i$ can be written in a matrix representation that is strictly a diagonal matrix of the form

$$M_i = \Delta \begin{pmatrix} \sqrt{f(x_1|z_i)} & 0 & 0 & 0 \\ 0 & \sqrt{f(x_2|z_i)} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \sqrt{f(x_N|z_i)} \end{pmatrix}.$$

Note that the matrices $M_i$ are strictly full-rank positive diagonal matrices and invertible. The expected value of $z$ (defined as <z>) is now shown to be the expected value of $|\hat{\Psi}(x,t)\rangle$ [228]:

$$\begin{aligned} < z > &= \int_{-\infty}^{\infty} z\, p(z) dz \\ &= \int_{-\infty}^{\infty} z\, r(z) dz \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} z \ \ |\phi(x, t_h, z_i)\ |^2 dz\, dx \\ &= \int_{-\infty}^{\infty} |\hat{\psi}(x, t_h)|^2 \int_{-\infty}^{\infty} z f(x|z) dz dx \\ &= \int_{-\infty}^{\infty} |\hat{\psi}(x, t_h)|^2 x\, dx \\ &= < x >. \end{aligned}$$

This shows that the value of $z$ that is expected is the value of $x$ that is expected. This result relates the statistics of $z$ to statistics of the wave function.

*Born Rule*

On any single trial, GRW has been proven to be invertible, but is clearly not distortionless. However, it is now proven that in terms of the average distribution of the square of the wave function, the GRW operation is in-fact an identity operator. This is also related to the common interpretation of the Born rule [6, p. 152] for which a measurement occurs with probability related to $|\hat{\psi}(x, t_h)|^2$, where $\hat{\psi}$ is the unitarily evolved wave function of the system at time $t_h$, i.e. given that no hits have yet occurred. If one considers $\psi(x, t_h)$ to be the wave function after GRW is applied, which represents the outcome or pointer of a measurement device, then in terms of the common interpretation of Born's rule one would desire that the average of $|\psi(x, t_h)|^2$, which will be denoted as $\overline{|\psi(x,t_h)|^2}$ is given by $|\hat{\psi}(x,t_h)|^2$.

We define $f(|\psi(x)\rangle|z=z_h)$ as the probability of finding the new state $\psi$ at $x$, that is after a GRW Gaussian centered at $z=z_h$ is multiplied by the state:

$$f(|\psi(x)\rangle|z=z_h) = |\psi(x,t_h|z=z_h)|^2$$
$$= \frac{|\phi(x,z_h)|^2}{r(z_h)}.$$

Weighting this by the probability of choosing $z_h$ and integrating over $z_h$ it is seen that $f(\,|\psi(x)\rangle) = \overline{|\psi(x,t_h)|^2}$ that is, the overall average probability distribution function of $|\psi(x,t_h)|^2$ after GRW is applied. Now,

$$f(|\psi(x)\rangle) = \int f\big(\,|\psi(x)\rangle|z=z_h)\big)\,f(z=z_h)dz_h$$
$$= \int \frac{|\phi(x,z_h)|^2}{r(z_h)}\frac{r(z_h)}{\int r(z)dz}dz_h$$
$$= \int |\phi(x,z_h)|^2 dz_h$$
$$= \int_{-\infty}^{\infty} f(x|dz_h)|\hat{\psi}(x,t_h)|^2\,dz_h.$$
$$= |\hat{\psi}(x,t_h)|^2 \int_{-\infty}^{\infty} f(x|z_h)\,dz_h$$

which gives when $f(x|z_h)$ has a Gaussian form

$$f\big(\,|\varphi_P\rangle(x)\big) = |\hat{\psi}(x,t_h)|^2.$$

This shows that in terms of the average distribution of the wave function, GRW is an identity operation in terms of the square of the wave function.

Such an identity relationship has been generalized to POVM operators in [229]. These authors consider a POVM with positive Hermitian operators $\{A_i\}$ with the property $\sum A_i = I$ where $i \in \Omega$ represents an outcome of the experiment and utilize the framework of *quantum instruments*. The POVM is implemented via a particular instrument that maps a given input state to an unnormalized output state given by $\mathfrak{I}_i(\varrho)$ such that $\|\mathfrak{I}_i(\varrho)\| = \mathrm{Tr}[\varrho A_i]$. An example of an instrument that implements a POVM $\{A_i\}$ is the Lüder instrument define via the relation $\mathfrak{I}_i(\varrho) = A_i^{1/2}\varrho A_i^{1/2}$. There is generally not a unique instrument for a given POVM $\{A_i\}$. The density matrix averaged over all $i \in \Omega$ is given by

$$\mathfrak{I}_\Omega(\varrho) = \sum_i \mathfrak{I}_i(\varrho).$$

A first kind measurement is defined in [229]. An instrument $\mathfrak{I}_i$ that implements a POVM $\{A_i\}$ is a first kind measurement if

$$\mathrm{Tr}[\mathfrak{I}_\Omega(\varrho)A_i] = \mathrm{Tr}[\varrho A_i], \forall \varrho, i \in \Omega.$$

for which the measurement statistics are identical for both the original state $\varrho$ and the state after measurement $\mathfrak{I}_\Omega(\varrho)$. Note that such a measurement may very well disturb the state on any particular outcome but does not disturb the statistics. There exists an adjoint map of $\mathfrak{I}_\Omega(\varrho)$ that maps bounded operators to bounded operators, denoted $\mathfrak{I}_\Omega^*(B)$ such that the probabilities are maintained $\mathrm{Tr}[\mathfrak{I}_\Omega(\varrho)A_i] = \mathrm{Tr}[\varrho \mathfrak{I}_\Omega^*(A_i)]$. A first kind measurement is also equivalent to the condition that the $A_i$ be fixed points of the adjoint map, i.e.,

$$\mathrm{Tr}[\mathfrak{I}_\Omega^*(A_i)] = A_i, \forall i \in \Omega.$$

*Criticisms of GRW*

There have been a number of criticisms of the GRW theory. Energy and momentum conservation are violated. There is a problem when one attempts to absolutely localize matter. If one does not absolutely localize matter, there are other problems due to the inherent tails of the matter that must exist. These issues are now discussed in more detail.

Violation of Energy and Momentum Conservation Laws

There is a substantial problem in attempting to localize a pure matter wave function. Consider an electron with an initially pure wave function that has some spread $\Delta x$. The energy of (a stationary) electron is related to the spread $\Delta x$. If $\Delta x$ were extremely small, the energy of the electron would be typically higher than, for example, a stationary electron that is widely spread. Hence providing a mechanism that affects the pure state of a matter wave function could easily become problematic in terms of conservation of energy and momentum.

Generally, the energy of a matter wave function changes when multiplied by a Gaussian function as in Equation (4.24). As shown in [230] energy can increase without bound and diverge with time. The authors propose a modification for which the energy increases asymptotically to a finite value rather than an infinite value. However, even with this modification, the model continues to violate energy conservation. Energy (and momentum) conservation is in our view a major problem with GRW that has not been resolved to-date. However, not all see this as a major problem. Violations of energy conservation that might be observed as a function of parameters of GRW has been further considered in [231].

In [232], Pearle proposes that there is a white noise background that can be incorporated into the Hamiltonian of the particles. The energy that is used to increase the particles energy in the process of reduction comes from a decrease in the noise background. If the white noise background is from external particles with mass, then if the system was sufficiently isolated in a vacuum chamber and experiments performed at sufficiently low temperature, then one would expect such noise sources could be reduced or eliminated. If the white noise background is from external radiation, then one could employ a Faraday cage to reduce these effects.

On the other hand, it has been suggested in [233] that there could be a fluctuating noise component in the gravitational metric which cannot be screened by a Faraday cage. More work is needed to confirm the claim of the existence of a physical white noise background that exists to balance energy in GRW collapse. A step in this direction has been made when considering the validation of the continuous version of GRW known as continuous spontaneous localization (CSL), which will be introduced shortly. The CSL parameters required to form a latent image are estimated in [234]. This work indicates that the original parameters of the noise coupling that were expected by GRW were too small, and a larger value is required. In the paper [235], high accuracy measurements of cantilever thermal fluctuations revealed a nonthermal force noise of unknown origin. This excess noise is stated to be compatible with the CSL noise predicted by Adler. With improving experimentation, this issue is expected to be resolved.

No-Tail Energy Conservation Problems

If one were to localize a particle in the classical sense, then the particle should be absolutely localized. If one attempts to localize an electron wave function to a single point, the wave function is fully localized but the energy of the electron is infinite as the momentum uncertainty will broaden to levels that are experimentally known not to occur in the measurement process. McQueen states regarding such localization [236],

> *See the print edition of The Quantum Measurement Problem for quotation.*

McQueen claims that in order to avoid energy increases associated with absolute localization of matter wave functions, GRW employed a Gaussian wave function to create a hit which has infinite tails, but such tails fall-off rapidly:

> *The relationship between energy and momentum then yields drastic post-collapse violations of energy conservation: ones that we know by experiment do not occur. So GRW formulated the collapse function as a Gaussian. The collapse then raises the mod-square of the chosen coefficient - the collapse centre - close to one while reducing the mod-square of all other coefficients close to zero but never actually to zero. GRW carefully chose the probability per unit time for spontaneous collapse and the width of the bell curve of the Gaussian ... so that the energy conservation violations are consistent with known experiments.*

## Causality and Hegerfeldt's Theorem

No absolute localization of a positive energy particle can occur without developing tails due to Hegerfeldt's theorem [237]. Hegerfeldt proved that any positive energy particle that is absolutely localized will immediately upon unitary evolution develop instantaneous tails. From Hegerfeldt's theorem, one cannot absolutely localize a positive energy particle such as an electron without violating causality.

Although approximate solutions are known [238], tails are generally a requirement of any positive energy solution. The fastest rate that the tail of a photon that is not absolutely localized can fall has been investigated by [239] and found to be exponential. The exponential falloff of a photon wave function can be of the order $e^{-f(r)}$where $f(r)$ increases with $r$ slower than linearly. Note that Gaussian tails fall faster than exponentially. It was further shown in [240] that the tails of free particles such as electrons that satisfy the Dirac equation fall faster than exponentially but cannot be strictly positive energy electrons and require a superposition of an electron and positron for representation. A similar result appears in [241], which appears to strictly rule out Gaussian tails. GRW would seem to necessitate the production of positrons if it is applicable to electrons.

## Tail problems

As discussed, when faced with either unphysical energy when absolute localization is employed, or finite energy when a tail is employed, McQueen states that GRW chose to have tails. Additionally, tails are needed to avoid Hegerfeldt causality violations associated with absolute localization.

The use of a wave function with an infinite tail may eliminate the large energy associated with absolute localization; however, the issue now becomes whether or not the use of wave functions with tails have problems of its own. Consider the argument put forward by Albert and Loewers in [242]. The authors consider a modification of Schrödinger's cat in the following manner. Consider a cat initially in the state $|R\rangle$ that observes an electron's spin in a given basis. If the electron spin is spin-up $|\uparrow\rangle$, unitary evolution is designed to allow the cat to live and evolve to state $|\text{Alive}\rangle$ which will be assumed to be in Position $x_1$ but if the electron is spin-down $|\downarrow\rangle$, the unitary evolution kills the cat which evolves to state $|\text{Dead}\rangle$ at Position $x_2 \neq x_1$. Now if the overall evolution were truly Schrödinger unitary, then if the electron is started initially in the superposition state $(|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$, the overall state $|\psi\rangle$ of cat and electron would evolve to:

$$|\psi\rangle = (|\text{Alive}\rangle \otimes |\uparrow\rangle + |\text{Dead}\rangle \otimes |\downarrow\rangle)/\sqrt{2}.$$

Now suppose GRW is applied to $|\psi\rangle$ which consists of a superposition of a cat at locations $x_1$ and $x_2$ by means of choosing a Gaussian that is centered at $z_h$. Let us assume that $z_h$ is approximately $x_1$. Then the state above will reduce to

$$|\psi\rangle = (\sqrt{1-a}\,|\text{Alive}\rangle \otimes |\uparrow\rangle + \sqrt{a}\,|\text{Dead}\rangle \otimes |\downarrow\rangle)/\sqrt{2}.$$

However, since measurement in GRW occurs in the preferred basis of position and GRW utilizes a Gaussian distribution which has infinite tails, it means that the cat which is in the state $|\text{Dead}\rangle$ will always have some amount $\sqrt{1-a}$ in which the tail continues to populate the state $|\text{Dead}\rangle$, i.e., while $a$ can be nearly zero, $a$ must remain strictly greater than zero. Albert and Loewer state in [242]

*See the print edition of The Quantum Measurement Problem for quotation.*

GRW appear to have skirted some (but not all) of the issues of absolute localization in the prior sections by means of introducing infinite tails. However, now they have created another problem for which the cat is always in a superposition for which the measurement problem does not appear to be resolved.

Complete Reversibility

GRW has been shown to be invertible, hence completely reversible. Consider a photon that is in a superposition of two locations with a given phase. The complete absorption of the photon by either detector destroys the photon coherence in the von Neumann theory. One would therefore expect that there should be an element of irreversibility to the theory that resolves the measurement problem. Yet complete reversibility is in principle possible in the GRW POVM given the POVM outcome. This doesn't prove that the theory is incorrect, but invertibility does not appear to be a strong point of GRW theory.

*Validating GRW*

Localization theories have the potential to address the requirements that have been required of a Category 1 Theory, R1.1-R1.4. In addressing R1.1, the theory predicts specific physical conditions or configurations under which non-Schrödinger evolution occurs versus Schrödinger evolution. GRW theory employs the two parameters $\lambda_h$ and $\sigma_h$ which specify the frequency of the GRW stochastic hits as well as the localization that occurs when the wave function is multiplied by a Gaussian of standard deviation $\sigma_h$. Hence when there are no hits, GRW predicts unitary evolution and when there are hits, the theory predicts non-Schrödinger evolution. Hence the theory partially meets requirement R1.1 as the parameters $\lambda_h$ and $\sigma_h$ are unknown.

In addressing R1.2, experiments can be used to verify that requirement R1.1 is met or violated. Firstly, one can determine what values of parameters $\lambda_h$ and $\sigma_h$ can be considered to be eliminated by results of known experiments. In the paper [243], the authors examine a number of experiments in relation to these parameters. The authors conclude that $\lambda_h$ should be greater than $10^{-17}$ $s^{-1}$ (termed the conventional value) and assume $\sigma_h$ is on the order of $10^{-5}$ cm. Testing this value would require an experiment that can diffract molecules 1000000 times larger than fullerenes. Assuming that the interaction of a particle with photographic film constitutes a bona fide measurement, this would require that $\lambda_h$ should be at least $10^8$ times larger than the conventional value. Testing this value would require an experiment that can diffract molecules 100 times larger than fullerenes. Other possible experiments include cantilevers. Due to the lack of experimental evidence, R1.2 can only be considered to be very partially fulfilled.

Requirement R1.3 is addressed, as GRW provides a nondeterministic prescription for the average change of state of the system, given that a GRW hit occurs. It is specified in [226] that the measurements are in a preferred basis which is that of localized states. And as well, GRW theory does provide a mechanism for the change of the state to a particular state via the nondeterministic Brownian motion. Hence GRW largely meets requirement R1.3. In terms of R1.4, the mechanics of both GRW and CSL reduce the state to a new state that agrees with the statistics associated with the von Neumann measurement postulate. However, the detailed mechanics of this change predicted either in GRW or in CSL have not been confirmed or seen experimentally, and as such have not yet met R1.4.