Rosenfeld's Solution

A unitary methodology was presented by Rosenfeld in 1965 [117] as an exposition of a measurement theory by Daneri, Loiniger, and Prosperi, based on the ergodic theorem [118]. This method uses a large number of device states and Rosenfeld believed that in the limit of large numbers of particles his method would be sufficient to explain measurement. Rosenfeld considers an atomic system initialized to state $|\varphi_0\rangle = \sum_r c_r|\varphi_r\rangle$ and a device initialized to state $|\Psi_0\rangle$. Two devices are consequently used to measure two different observables *A*, *B* with eigenstates given by $|\varphi_r\rangle$ and $|\chi_s\rangle$ respectively. There is a short time τ when the device interacts with the system, such that if the initial state of the system is $|\varphi_r\rangle$ the final state of system-device is $\alpha_r|\varphi_r\rangle\otimes|\Psi_r\rangle$, whereby the device is in the pointer state $|\Psi_r\rangle$ that occurs unitarily when the system input is in state $|\varphi_r\rangle$. The approximation $\alpha_r \approx 1$ is made as the interaction is assumed to be sufficiently short. After time τ, the atomic system and device are assumed non-interacting and evolve via the tensor product unitary $U_S(t) \otimes U_D(t)$. The Schrödinger predicted unitary evolution when the device is set to measure observable *A* is given by

$$\sum_r c_r \, |\varphi_r\rangle \otimes |\Psi_r\rangle.$$

After a time t the system-device evolves to

$$\sum_r c_r \, U_S(t)|\varphi_r\rangle \otimes U_D(t)|\Psi_r\rangle. \tag{4.4}$$

The system is then followed by a measurement of Observable *B* via interaction with a second measurement device, initialized to $|\Psi_0\rangle$, that evolves to $|\chi_s\rangle\otimes|\Psi_r\rangle$ when the system input is in state $|\chi_s\rangle$. The probability of finding the quantum state $|\chi_s\rangle$ is on average given by use of the Born and/or von Neumann measurement postulate

$$\sum_r |c_r|^2 \langle\chi_s|U_S(t)\varphi_r\rangle^2. \tag{4.5}$$

Rosenfeld states that since this expression only depends on the relative probabilities $|c_r|^2$ of the eigenstates $|\varphi_r\rangle$, but not on the phases of the complex coefficients c_r, this is referred to as the "reduction" of the state $|\varphi_0\rangle$. That is, if no measurement of *A* had been performed, the probability of finding $|\chi_s\rangle$ is given by

$$|\sum_r c_r \, \langle\chi_s|U_S(t)\varphi_r\rangle|^2 \tag{4.6}$$

which differs from Equation (4.5) by the cross terms:

$$\sum_{r\neq r'} c_r \, c_{r'}^* \langle\chi_s|U_S(t)\varphi_r\rangle\langle\chi_s|U_S(t)\varphi_{r'}\rangle^*. \tag{4.7}$$

Rosenfeld claims that the effect of Measurement *A* is the disappearance of the terms in Equation (4.7) and that is formally denoted as the "reduction" of the initial state.

Rosenfeld desires to impose an ergodicity condition on the measurement device for which Equation (4.7) is claimed to go to zero when the device is macroscopic. In this case, if Measurement *A* is followed by Measurement *B*, the probability of finding $|\chi_s\rangle$ will correspond correctly to Equation (4.5). The argument Rosenfeld presents is given in the Appendix.

External orthogonalization will also be useful in discerning many approaches that have been proposed to resolve the measurement problem.

External Orthogonalization and Rosenfeld

Rather than use the method of ergodicity that is statistical and outside of a rigorous Hamiltonian formulation that has been the focus of this book, it will be shown that external orthogonalization is a unitary solution within the Hamiltonian formulation, that can also be used to derive Equation (4.5).

Consider the use of two unitary devices denoted *X* and *Y* that are used to implement external orthogonalization in order to determine if such devices result in the outcome that Rosenfeld desires under measurement, Equation (4.5). In the first measurement *X* of Rosenfeld, let us set the eigenstates of the system required by external orthogonalization equal to the eigenstates of Observable *A* in Rosenfeld's proposal, i.e., $|\phi_r\rangle|\varphi_r\rangle$ and whereby the corresponding unitary evolution is denoted $U_{S,X}(t)$. The second measurement is specified by using external orthogonalization with $|\phi_r\rangle = |\chi_r\rangle$, the eigenstates of Observable *B*, and the corresponding unitary evolution is denoted $U_{S,Y}(t)$. In both cases the external particles are initialized to $|\Psi_0\rangle$. Given the initial state $|\psi_S\rangle = \sum_r c_r|\phi_r\rangle$, the resulting state of external orthogonalization after unitary evolution is

$$|\psi_{S,X}^{(1)}\rangle \equiv U_{S,X}(\pi/2)(|\psi_S\rangle \otimes |\Psi_0\rangle)$$

$$|\psi_{S,X}^{(1)}\rangle = \sum_r c_r|\varphi_r\rangle \otimes |\Psi_r\rangle.$$

Rosenfeld then proposes that the system and Device *X* are separated or non-interacting and the system is allowed to unitarily evolve via its self-Hamiltonian. In this case, consider the partial density matrix of the system. The system is in a mixed state, $\rho_S^{(1)} \equiv \mathrm{Tr}_X |\psi_{S,X}^{(1)}\rangle\langle\psi_{S,X}^{(1)}|$ which can be computed as the diagonal matrix

$$\rho_S^{(1)} = \begin{pmatrix} |c_1|^2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & |c_N|^2 \end{pmatrix}$$

where *N* is the number of eigenvectors of Observable *A*. After the system and Device *X* are separated, the system is evolved unitarily for a time $t = t_U$. Denoting

$$U_S(t) = e^{-\frac{iH_S t}{2}}$$

the resulting state of the system becomes $\rho_S^{(2)} \equiv U_S(t_U)\rho_S^{(1)}U_S(t_U)^\dagger$. This will leave the density matrix unchanged i.e., $\rho_S^{(2)} = \rho_S^{(1)}$ as $\rho_S^{(1)}$ is diagonal in the self-Hamiltonian H_A. The third and final step is to interact the system (specified by density matrix $\rho_S^{(2)}$) with Device Y initialized to the density matrix $\rho_{D,0} \equiv |\Psi_0\rangle\langle\Psi_0|$. The density matrix of System-Device Y is then given by

$$\rho_{S,Y}^{(3)} \equiv\ U_{S,Y}(\pi/2)(\rho_S^{(2)}\otimes \rho_{D,0})U_{S,Y}(\pi/2)^\dagger\ .$$

The system after interacting with Device Y is given by $\rho_S^{(2)} \equiv \mathrm{Tr}_Y|\psi_{S,Y}^{(1)}\rangle\langle\psi_{S,Y}^{(1)}|$, which can be computed as

$$\rho_S^{(2)} = \begin{pmatrix} \sum_r |c_r|^2\langle\chi_1|\varphi_1\rangle^2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sum_r |c_r|^2\langle\chi_N|\varphi_N\rangle^2 \end{pmatrix},$$

and is in agreement with Rosenfeld's Equation (4.5). Hence it has been shown that unitary evolution via external orthogonalization in the manner that Rosenfeld considered, is fully consistent with Born's rule and von Neumann's measurement postulate insofar as results on the system are measured.

External Orthogonalization and UMDT

If the measurement problem could be resolved by developing a unitary approach that results in Rosenfeld's Equation (4.5), then external orthogonalization would indeed be a satisfactory resolution to the measurement problem. If this were the case, then external orthogonalization should also be able to satisfactorily resolve the dilemma for which the Chapter 3 UMDT was specifically developed to address. Let us then examine how well Rosenfeld's criterion and external orthogonalization fare in comparison with the UMDT.

Suppose that the external orthogonalization Hamiltonian of Equation (4.1) is utilized in the Chapter 3 development for the detectors shown in Figure 3.4. Consider the case for which both devices are identical and are two-level systems in which $|\Psi_0\rangle$ that will change its state to $|\Psi_1\rangle$ upon absorbing a photon. Setting $|\phi_0\rangle\ = |0\rangle$ and $|\phi_1\rangle\ = |1\rangle$, a Hamiltonian applicable to Figure 3.4 is given by:

$$H = H_F\ \otimes I \otimes I + I \otimes H_1 \otimes I + I \otimes I \otimes H_2 + H_{int}$$

$$H_F = wa_B{}^\dagger a_B \otimes I_2 + wI_2 \otimes a_C{}^\dagger a_C,$$

$$H_i = |0\rangle\langle 0| + 2|1\rangle\langle 1|$$

$$H_{F,1} = a_B \otimes I_2 \otimes \sigma_+ \otimes I_2 + a_B{}^\dagger \otimes I_2 \otimes \sigma_- \otimes I_2$$

$$H_{F,2} = I_2 \otimes a_C \otimes I_2 \otimes \sigma_+ + I_2 \otimes {a_C}^\dagger \otimes I_2 \otimes \sigma_-$$

$$H_{int} = H_{F,1} + H_{F,2}.$$

In this model, the atomic system that is considered to be measured in Rosenfeld's paper has been replaced by a photon in order to consider its application using the Chapter 3 development. Both H_1 and H_2 are device Hamiltonians that are each a two-level system. The field is described by a harmonic oscillator of mode energy $w = 1$, and the interaction is such that when a single photon is annihilated the device resonantly undergoes a raising operation and when the device falls from state $|1\rangle$ to $|0\rangle$ a resonant photon is added to the mode. The application of this particular external orthogonalization Hamiltonian can be seen to be a 100% absorbing model for interaction times $\pi/2$ (and modulus 2 π).

The initial state of the device i, denoted $|\psi_r^0\rangle_i$, unitarily evolves to a read-out state that is indicative of no measurement and, adopting the notation in Chapter 3, is denoted $|\psi^0\rangle_i,\ i = 1, 2$. Consider the case that a photon with state $|1_V\rangle_B$ is inserted in Path *B*. After time $\pi/2$, the state of Device 1 evolves to its read-out state indicating that the Device has detected a photon, denoted $|\psi^1\rangle_1$. Similarly, if a photon with state $|1_H\rangle_C$ is inserted solely in Path *C*, the final state of Device 2 after time $\pi/2$ is denoted as $|\psi^1\rangle_2$.

In the case of Rosenfeld whereby a single device is used in order to detect the system, the initial system state was assumed to be a pure state given by $|\psi_S\rangle = \sum_r c_r\, |\varphi_r\rangle$. However, in the Chapter 3 development there are now two detectors that interact with the electromagnetic field in separate locations. However, for the case needed to be considered in Figure 3.4, the initial photon state given by

$$|\psi_{photon,PBS}\rangle = \sqrt{a}|1_V\rangle_B \otimes |0\rangle_C + \sqrt{1-a}\,|0\rangle_B \otimes |1_H\rangle_C$$

results in a mixed photon state for both photon Mode *B* as well as photon Mode *C*. In the case of Figure 3.4 there are two detectors after the initial Step 1 unitary evolution and the partial density matrices of the devices are found by Equation (3.32) for which $\rho_{1'}^{(1)} = a\rho_{1,1'}^{(1)} + (1-a)\rho_{2,1'}^{(1)}$ and $\rho_{1,1'}^{(1)}$ is the Schrödinger evolved density matrix of the device assuming a photon impinges on the device and $\rho_{2,1'}^{(1)}$ is the Schrödinger evolved density matrix of the device with no photon impinging on the device. Assuming for simplicity that the initial states of the devices are pure states, then $\rho_{1,1'}^{(1)}$ and $\rho_{2,1'}^{(1)}$ are pure orthogonal states. With this, if one considers $\rho_{1'}^{(1)}$ transformed via change of basis using the final orthogonal states $\rho_{1,1'}^{(1)}$ and $\rho_{2,1'}^{(1)}$, the result is

$$\begin{pmatrix} a & 0 \\ 0 & 1-a \end{pmatrix} \tag{4.8}$$

and $\rho_{2'}^{(1)}$ can similarly be expressed in the basis of the final orthogonal states $\rho_{1,2'}^{(1)}$ and $\rho_{2,2'}^{(1)}$ resulting in

$$\begin{pmatrix} 1-a & 0 \\ 0 & a \end{pmatrix}. \tag{4.9}$$

Hence the devices after Step 1 are in a mixed state for which the diagonal elements correspond exactly to what is required in the measurement postulate and Born's law. If the solution to the measurement problem only required that one show how one could derive the probabilities of measurement via unitary evolution, then Step 1 would suffice and the measurement problem would be solved by Step 1. However, as we have defined the measurement problem, the problem is related to the unitary prediction of entanglement versus a product state of the device which is expected under measurement. This is brought out in the UMDT operations that distinguish unitary evolution from measurement, and there remain Steps 2 and 3. If one applies Step 2 and Step 3 to the two devices, the result is shown in Figure 3.16. Although the partial density matrix of the devices after Step 1 by themselves are in states that might appear to resolve the measurement problem as understood by Rosenfeld and many others, one sees in Figure 3.16 that the devices when acting unitarily have the Bell measurement result that is always greater than the result when measurement occurs for any non-zero superposition parameter a.

Let us now ask how well external orthogonalization solves the measurement problem by examining the requirements that we have set forward for a scientifically acceptable solution. It has been demonstrated using the UMDT of Chapter 3, that reproducing Rosenfeld's Equation (4.5) is not sufficient to resolve the measurement problem. The measurement problem is related to the prediction of entanglement versus the product state that is demanded in any theoretical treatment. That Schrödinger's equation predicts entanglement was pointed out quite early in 1935 by Schrödinger himself [119].

In order to show that the measurement problem can be solved unitarily, one would need to develop a unitary methodology that gives the Born predictions of measurement and for which the device plus system are in a tensor product state versus an entangled state. The fact that the partial density matrices of the devices are in completely mixed states, when the overall state of the devices is a pure state, can be shown to result in an entangled state (see Exercise 4.4 in the book or kindle version of theQMP). Hence the idea that a unitary operation can resolve the measurement problem by diagonalizing the density matrix of the devices in a manner related to Born's law, such that it removes first order coherence, is not correct. On the contrary, the fact that the partial density matrices are mixed and non-coherent when the overall joint matrix of the two density matrices is pure, only points to the transfer of the first order photon coherence that originally existed between the two paths into second order coherence of entanglement between the two qubits via the UMDT in Step 1 and 2. Such entangled states are contrary to the states that result during measurement, which are well-defined product states. The unitary prediction of entanglement is contrary to the notion of a single well-defined system-device state that occurs under the measurement process.

Has Rosenfeld addressed Requirement R1.1? There are not explicit conditions that have been specified for a measurement to occur. Rosenfeld has not addressed R1.1 as he contends that measurement is a consequence of Schrödinger unitary evolution. And if the measurement problem could be resolved by showing that Equation (4.5) converges to Equation (4.6), then indeed Rosenfeld could have resolved the measurement problem.