Master Equations for Deterministic Evolution

A quantum master equation of the deterministic Lindblad form is a deterministic equation for which a system interacts with a larger system such as a thermodynamic bath for which the overall evolution is often taken to be unitary. Another type of master equation is a stochastic master equation for which the overall evolution is generally non-unitary. Stochastic Schrödinger equations will be considered later in this chapter for which the average density matrix obeys a Lindblad equation.

Consider a Hamiltonian consisting of a system and an environment of the form

$$H = H_S \otimes I + I \otimes H_E + H_{S,E} \tag{4.20}$$

where H_S is the self-Hamiltonian of the system, H_E is the self-Hamiltonian of the interacting environment and $H_{S,E}$ is the interaction Hamiltonian between system and environment. We denote $\mathfrak{M}(\mathcal{H})$ as the set of density matrices in a Hilbert space $\mathcal{H}$, the system Hilbert space as $\mathcal{H}_S$ with density matrix $\rho_S \in \mathfrak{M}(\mathcal{H}_S)$ and the system-environment Hilbert space by $\mathcal{H}_{S,E}$ and system-environment density matrix $\rho_{S,E} \in \mathfrak{M}(\mathcal{H}_S \otimes \mathcal{H}_E)$. When the overall evolution of the system and interacting environment is unitary the evolution of the system density matrix ρ_S is generally non-unitary and can be computed by tracing out the environmental degrees of freedom from the joint density matrix $\rho_{S,E}$ as follows:

$$\rho_S = \mathrm{Tr}_E[\rho_{S,E}], \tag{4.21}$$

where Tr_E denotes the operation of tracing the environment from the joint density matrix $\rho_{S,E}$. It can be proven that in the case where the system is coupled to a Markovian environment, the trace operation results in the following form [157]:

$$\frac{d\rho_S}{dt} = -\frac{i}{\hbar}[H_S, \rho_S] + \gamma \sum_j \left(L_j \rho_S L_j^\dagger - \left(\frac{1}{2} \rho_S L_j^\dagger L_j + L_j^\dagger L_j \rho_S \right) \right). \tag{4.22}$$

This is the most general form of Markovian master equation describing evolution that is trace-preserving and completely positive. Note that the first term on the right-hand side of Equation (4.22) is the Schrödinger equation for ρ_S in the form of a density matrix, which would be the unitary evolution of the system in absence of system-environmental coupling. The second term of the right-hand side of Equation (4.22) specifies non-unitary evolution with respect to H_S in a manner that maintains the unitary evolution of $\rho_{S,E}$ with respect to the overall Hamiltonian of Equation (4.20). As the deterministic Lindblad master equation is non-unitary, it makes sense to examine whether or not such equations are useful toward resolving the measurement problem.

In this respect, let us examine what the deterministic Lindblad equation would predict regarding the UMDT of Chapter 3. Consider as before an apparatus that has a latency time τ_l for registration and that the uncertainty of the time of emission of the photon is Δt_e. It was assumed in Chapter 3 that one can separate the two devices in Figure 3.4, via an arbitrary large distance d so that $d > 2c(\Delta t_e + \tau_l)$. In this case, one only has to include the particles of the environment that are within a time-like window, or those that are within a distance of $c(\Delta t_e + \tau_l)$. If one were to model the particles via a deterministic Lindblad equation, there can be no interaction between the two separated devices. Interaction can only occur between the *B* mode of the photon and Device *1* and as well between the *C* mode of the photon and Device *2*. This is precisely the setup of the UMDT of Chapter 3 for which it has already been proven that unitary evolution that includes the local environment within the time-like window $d < c(\Delta t_e + \tau_l)$ will always produce the CHSH sum of $2\sqrt{2}$. Granted, one could restrict the application of the UMDT test of Chapter 3 to the system only. In this case the system is in a diagonal state as found in Equation (4.3) reproduced below for convenience,

$$\begin{pmatrix} |c_1|^2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & |c_N|^2 \end{pmatrix}.$$

The UMDT test of Chapter 3 if applied only to the system would yield the CHSH sum less than the value *2* since, even unitarily, there would be no direct entanglement between the two system subspaces without including the Devices and their local environments. However, one cannot conclude from a UMDT test on the system alone that the system is in a particular or pure state for which one of the *N* possibilities has indeed occurred. The Chapter 3 UMDT test on the system alone will always produce a value less than or equal to 2, whether or not interaction is unitary or occurs via the measurement postulate. Hence, applying the test to only the system provides no experimental discriminating power to address the hypothesis test in Chapter 3. In order to have discriminating power, one must include the system, Device *1* and its local environment, for which the CHSH sum of $2\sqrt{2}$ implies that entanglement exists, and the system is *necessarily* in the mixed state of Equation (4.3). That is, the system is not a pure state that would be indicative that a particular outcome had occurred.

Deterministic Lindblad equations are a useful tool to compute the reduced density matrix of the system. If one assumes that the initial state of the devices and surrounding environment are in a pure state, it is known that unitary evolution maintains the purity of the state, i.e., the overall composite state remains pure under unitary evolution. In this case the finding of the system being in a mixed state such as in Equation (4.3) indicates, at least theoretically, that the system is further entangled with other degrees of freedom. That is, a bipartite system that has an overall pure state with subsystems that are in a mixed state indicates (at least theoretically) that the system is further entangled with the environment. As a deterministic Lindblad equation would lead to a mixed state description of the system similar to Equation (4.3) the experienced quantum information theorist would confirm our dictum that:

To the extent there is entanglement, there is no measurement.