Master Equations for Nondeterministic Evolution

Master equation approaches have been developed that provide an equation for the density matrix of a reduced system that interacts with a larger system such as the environment [258]. In 1976 both Gorini, Kossakowski, and Sudarshan (GKS) for the case of discrete systems [249] and Lindblad [250] for the case of continuous systems, developed the most general Markovian master equation that is trace-reserving and completely positive. The Lindblad form for the reduced system dynamics has been shown to be equivalent to tracing out the unitary interaction of a system with a Markovian environment. For example, it has been shown in [157] using the Jaynes-Cummings model as the interaction between a system and oscillator bath, that a Lindblad form results for the evolution of the density matrix ρ of the system. A general Lindblad form for the evolution of a reduced density matrix is given as

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H, \rho] + \sum_i \left(L_i \rho L_i{}^\dagger - \frac{1}{2} L_i{}^\dagger L_i \rho - \frac{1}{2} \rho L_i{}^\dagger L_i \right) \quad (4.34)$$

where L_i are operators that represent the interaction of the system and environment.

When starting in a pure state, the density matrix can evolve deterministically to a mixed state in a master equation. Although master equations such as the Lindblad form are deterministic equations in the evolution of the density matrix, one might nevertheless consider the use of stochastic (nondeterministic) differential equations to model the individual stochastic outcomes such that, on average, the stochastic states average exactly to the deterministic mixed state predicted in a master equation. In such a case when the individual trials are stochastic, the average system evolution may be described by a nondeterministic master equation.

Gisin and Percival in [259] have shown that the stochastic differential equation

$$\begin{aligned} d|\hat{\psi}_t\rangle \;=\; & -\frac{i}{\hbar} H|\hat{\psi}_t\rangle dt + \sum_i \left(\bar{L}_i L_i - \frac{1}{2} L_i{}^\dagger L_i - \frac{1}{2} \bar{L_i}{}^\dagger \bar{L}_i \right) |\hat{\psi}_t\rangle dt \\ & + \sum_i (L_i - \bar{L}_i) dB_t |\hat{\psi}_t\rangle \end{aligned} \quad (4.35)$$

has the mean density matrix given by

$$\rho = \overline{|\hat{\psi}_t\rangle\langle\hat{\psi}_t|}$$

where ρ satisfies the Lindblad form of Equation (4.34). This shows that one can effectively compute or simulate the Lindblad reduced density matrix of a system that interacts unitarily with an environment via the use of stochastic differential equations. One can also derive a Lindblad form for energy-driven reduction models [255].

Furthermore, note that the equation of the density matrix ρ in Equation (4.34) is linear in ρ. Hence the general evolution of such equations is non-linear in the wave function and linear in the density operator. We will see later in Chapter 7 that such nondeterministic evolution is within a class of time-evolution that does not suffer from problems of signaling.

The fact that stochastic equations of the form of Equation (4.35) have the property of quantum state convergence to one of the operator's range space in agreement with von Neumann theory, and furthermore that the evolution is non-linear in the wave function yet linear in the density operator, gives impetus to consider the possibility that stochastic equations of the form of Equation (4.35) govern the theory of measurement. This possibility will be further examined in Chapter 7. One might also consider as well that the ensemble average in terms of a density matrix corresponds exactly to the Lindblad equation. However, this has positive and negative consequences: it is positive as it appears to correspond to real-world effects in open-system theory. It is negative in the sense, as has been already discussed previously, that such master equations also follow from a unitary deterministic theory that does not resolve the measurement problem as a Category 1 theory. Hence to experimentally distinguish such a Category 1 theory from a purely unitary theory would require isolating or minimizing a system due to decoherence to rule out the possibility that a non-unitary effect is actually a unitary effect in a higher Hilbert space.