Stochastic Differential Equations

GRW theory as originally proposed has discrete time jumps, for which the wave function evolution is discontinuous. Pearle in 1973 suggested the use of stochastic differential equations to account for the reduction process [244]. With the use of a stochastic differential equation, changes do not occur discontinuously in a single jump, but can be made to change in a continuous manner. The heuristic idea of stochastic differential equations is to increase the rate of occurrence of the jumps while simultaneously decreasing the effect that the stochastic process has on the change on the state for any given occurrence. For example, in the case of GRW one might consider increasing the Poisson parameter λ_h so that in the limit, events are expected to occur continuously, while simultaneously decreasing the effect of any given hit. In this manner, a stochastic process can arise that is expected to be continuous in the manner that the state-vector changes, but still leads to the stochastic changes that are in agreement with the Born rule.

Stochastic differential equations often utilize continuous Brownian motion in their description. A stochastic process $\{B_t, t > 0\}$ is called a real Brownian motion if the following hold [245]:

1. The initial value B_0 is a real value (for a real Brownian motion process).
2. The process has independent increments, i.e. for all times $0 \leq t_1 \leq \cdots \leq t_N$ the increments $B_{t_N} - B_{t_{N-1}}, B_{t_{N-1}} - B_{t_{N-2}}, \cdots, B_{t_2} - B_{t_1}$ are independent random variables
3. For all t the increments $B_{t+\sigma^2} - B_t$ are normally distributed random variables $N(0, \sigma)$, i.e. with mean 0 and standard deviation σ.
4. Almost surely, the function $t \to B_t$ is continuous.

A standard Brownian motion $\{B_t, t > 0\}$ has initial value $B_0 = 0$. Brownian motion can be extended to a vector form of N dimensions. By considering a two-dimensional real Brownian process, a standard complex Brownian motion can be defined in which both real and imaginary components are normally distributed. We will use the convention in [246] for which a complex motion is defined as

$$B_t = \frac{1}{\sqrt{2}}\left(B_t^{(1)} + iB_t^{(2)}\right)$$

where $B_t^{(1,R)}, B_t^{(2,R)}$ are independent real standard Brownian motions. This definition has the advantage that B_t continues to have variance parameter σ^2. The Itô rule for this case is [247] $d\mathrm{B}_t' dB_t = dt, dB_t dB_t = 0$.

The development of stochastic differential equations was mathematically formalized by Itô and in a different form by Stratonovich. These two forms are based on different definitions and the use of one form over another has various advantages depending on the situation [185]. The theory of stochastic differential equations developed by Itô was extended in the seminar paper by Hudson and Parthasarathy

[248], which also relates to the work of [249] and [250].

We utilize the Itô form which can be written as an equation that has an added stochastic term of the form:

$$dx_t = \mu(x_t, t)dt + \sigma(x_t, t)dB_t, \tag{4.25}$$

where B_t is a complex Brownian stochastic process.

Denoting the change in x_t as Δx over a small interval of time Δt, it is seen that the distribution of Δx is given a normally distributed random variable with mean $\mu(x,t)\Delta t$ and variance $\sigma(x,t)^2\Delta t$. Moreover, as Δx can be written independently of the x, one sees that the process is a Markov process. The term $\mu(x_t, t)$ is referred to as the drift coefficient and $\sigma(x_t, t)$ as the diffusion coefficient. While B_t is a complex Brownian motion process, the stochastic process x_t is often referred to as a diffusion process.

In order to apply the Itô form to the evolution of a wave function $|\psi_t\rangle$ consider associating $|\psi_t\rangle$ with x above and restricting initially the evolution to the linear stochastic differential equation

$$d|\psi_t\rangle = C|\psi_t\rangle dt + A|\psi_t\rangle dB_t \tag{4.26}$$

where C , A are linear operators on $|\psi_t\rangle$. This form can be extended to a summation over multiple A_i, but for now we assume a single A_i. It is shown in [244] that unless further restrictions are put in place, $|\psi_t\rangle$ will generally have a norm that changes in time. It is found that the norm can be written as:

$$\begin{aligned}\langle\psi_t|\psi_t\rangle = \langle\psi_0|\psi_0\rangle + \int_0^t \langle\psi_s|(C + C^\dagger + A^\dagger A)\psi_s\rangle ds + \\ \int_0^t \langle\psi_s|(A + A^\dagger)\psi_s\rangle dB_s.\end{aligned} \tag{4.27}$$

If one further restricts the above so that the second term is zero via

$$C + C^\dagger = -A^\dagger A$$

then $\langle\psi_t|\psi_t\rangle = \langle\psi_0|\psi_0\rangle + \int_0^t \langle\psi_s|(A + A^\dagger)\psi_s\rangle dB_s$. Now if we set the anti-Hermitian part of C to $-\frac{i}{\hbar}H$, Equation (4.26) becomes

$$d|\psi_t\rangle = -\frac{i}{\hbar}H|\psi_t\rangle dt - \frac{1}{2}A^\dagger A|\psi_t\rangle dt + A|\psi_t\rangle dB_t\,. \tag{4.28}$$

The above equation is a linear equation in the states $|\psi_t\rangle$. In order to further restrict $|\psi_t\rangle$ so that it has constant unity norm, non-linear terms are introduced. Let

$$|\hat{\psi}_t\rangle = \frac{|\psi_t\rangle}{\big||\psi_t\rangle\big|}.$$

It is shown in [244] that this restriction results in the Itô equation

$$d|\widehat{\psi}_t\rangle = -\frac{i}{\hbar}H|\widehat{\psi}_t\rangle dt + \left(-\frac{1}{2}(A^\dagger - \overline{R})|\widehat{\psi}_t\rangle + \frac{1}{2}(A - \overline{R})\overline{R}|\widehat{\psi}_t\rangle\right)dt + (A - \overline{R})dB_t|\widehat{\psi}_t\rangle \tag{4.29}$$

where $\overline{R} \equiv \frac{1}{2}\langle\widehat{\psi}_t|A + A^\dagger|\widehat{\psi}_t\rangle$. Note that once the restriction is made to the normalized form $|\widehat{\psi}_t\rangle$, the resulting Equation (4.29) is now nonlinear in the states $|\widehat{\psi}_t\rangle$. Assuming self-adjoint operators $A = A^\dagger$ the above can be written as

$$d|\widehat{\psi}_t\rangle = -\frac{i}{\hbar}H|\widehat{\psi}_t\rangle dt - \frac{1}{2}(A - \bar{A})^2|\widehat{\psi}_t\rangle dt + (A - \bar{A})dB_t|\widehat{\psi}_t\rangle \tag{4.30}$$

where $\bar{A} \equiv \langle\widehat{\psi}_t|A|\widehat{\psi}_t\rangle$. This can be rewritten in the form

$$d|\widehat{\psi}_t\rangle = -\frac{i}{\hbar}H|\widehat{\psi}_t\rangle dt + \left(\bar{A}A - \frac{1}{2}A^2 - \frac{1}{2}\bar{A}^2\right)|\widehat{\psi}_t\rangle dt + (A - \bar{A})dB_t|\widehat{\psi}_t\rangle. \tag{4.31}$$

This can be further extended to a summation over multiple A_i which results in

$$d|\widehat{\psi}_t\rangle = -\frac{i}{\hbar}H|\widehat{\psi}_t\rangle dt + \sum_i\left(\bar{A}_iA_i - \frac{1}{2}A_i^2 - \frac{1}{2}\bar{A}_i^2\right)|\widehat{\psi}_t\rangle dt + \sum_i(A_i - \bar{A}_i)dB_t|\widehat{\psi}_t\rangle. \tag{4.32}$$

State Reduction

A result established by Gisin in [251] is that for a single adjoint operator A, the evolution of $|\widehat{\psi}_t\rangle$ converges (in the sense that the mean square deviation approaches zero) asymptotically to an eigenstate of A. In [226] conditions were found for the case of multiple operators A_i that the evolution of $|\widehat{\psi}_t\rangle$ moves into subspaces that are associated with the operators A_i in the stochastic differential equations. A similar result was established in [252] showing that the dispersion entropy, which is a measure of localization of $|\widehat{\psi}_t\rangle$ to one of several subspaces, generally decreases in time in a manner for which $|\widehat{\psi}_t\rangle$ converges to evolve into one of the subspaces.

A condition on the subspaces that appears to be met in both papers is described. Let us assume that the operators A_i in Equation (4.32) are Hermitian, and for simplicity assume that A_i can be represented by orthogonal projection operators P_i ($P_i^2 = P_i$, not necessarily rank 1, $P_i\,P_j = 0$ for all $i \neq j$). Let $\Re(P)$ denote the range space of an operator P. Note that if $x \in \Re(P)$, $P_ix = x$, and P_i is the identity map for all $x \in \Re(P)$.

An initial state $|\widehat{\psi}_t\rangle$ is found to evolve to $\Re(P_i)$ for some P_i. However, the

particular outcome i occurs stochastically. That is, under any given trial for which the stochastic differential equation is evaluated for all time, the evolution will converge into $\Re(P_i)$ for some particular P_i. This shows that such stochastic differential equations with Brownian motion converge to a single subspace, which is a prediction of standard von Neumann measurement theory. Asymptotic spectral stability has been proven in [253].

Furthermore, the probability of the state asymptotically belonging to one of the subspaces, i.e., $|\hat{\psi}_t\rangle \in \Re(P_i), t \to \infty$, is given by $\langle\hat{\psi}_0|P_i|\hat{\psi}_0\rangle$. Hence the stochastic differential equations with Brownian motion reproduce the quantum prediction of Born's rule which is also a prediction of standard von Neumann measurement theory. Quantum diffusion equations developed by Gisin and Percival were shown to be related to the work of Hudson and Parthasarathy in [254].

Although stochastic differential equations generally converge into eigenstates of observables, such equations often do not conserve energy or momentum. If the conditional expected value of the random variable representing the dynamic energy $H_t \equiv \mathrm{Tr}(\rho(t)H)$ has the martingale property $\mathrm{E}(H_t|\{H_\tau : \tau \leq s\}) = H_s$ for $s \leq t$ then energy will be conserved on average. Consider setting $A = H$ in Equation (4.30)

$$d|\widehat{\psi}_t\rangle = -\frac{i}{\hbar}Hd|\widehat{\psi}_t\rangle\,dt - \frac{1}{2}(H-\bar{H})^2|\widehat{\psi}_t\rangle dt + (H-\bar{H})dB_t|\widehat{\psi}_t\rangle. \tag{4.33}$$

Equation (4.33) has the desired martingale property, and stochastic equations of this form are defined in [255] [256] as energy-driven stochastic reduction models. As such, one might consider such a model for a closed system collapse reduction model. However, on-average conservation is not the same as strict energy conservation on every trial and, as will be further discussed in Chapter 5, was at the heart of the Bohr-Kramers-Slater debate. Additionally, it has been argued by [257] that energy-driven models cannot lead to spatially localized states in commonly occurring cases. Whether or not energy is strictly conserved or conserved on-average is an important question that will be further discussed in Chapters 5, 7, and 8.