Appendix

Details from The Fall of Classicality

In his model of cavity oscillators, Planck was able to show a simple relation between the radiated energy density and the energy U for a single oscillator is

$$B(\nu, \mathrm{T}) = \frac{8\pi\nu^2}{c^3} U(\nu, T). \tag{A.1}$$

The function $U(\nu, T)$ was not known; however, Planck took a hint at how to proceed from the fundamental thermodynamic relation between entropy and energy changes in terms of the absolute temperature:

$$dS = dU/T. \tag{A.2}$$

Planck was inspired by Equations (A.1) and (A.2) to guess $U(\nu, T)$ by interpolating the second derivative d^2S/dU^2 between the Rayleigh-Jean and Wien laws via the equation

$$-\left(\frac{d^2S}{dU^2}\right)^{-1} = \alpha U + \beta U^2 . \tag{A.3}$$

The second term in Equation (A.3) corresponds to the Rayleigh-Jeans formula which implies $T \propto U$ at low frequencies and thus $-(d^2S/dU^2)^{-1} \propto U^2$ using Equation (A.2). The first term corresponds to the Wien formula which implies $1/T \propto -\ln U$ and $-(d^2S/dU^2)^{-1} \propto U$ at high frequencies using Equation (A.2). Note that the full entropy obtained by integrating Equation (5.3) is of the form [412]:

$$S(U) = k\left[\log(1 + U/h\nu)^{1+U/h\nu} - \log(U/h\nu)^{U/h\nu}\right] \tag{A.4}$$

which suggested to Planck the combinatorial counting approach for the probabilities needed for Boltzmann's entropy formula with energies $E = h\nu$.

Einstein was able to show that high-frequency heat radiation carries a distinctive signature of finitely many spatially localized independent components. He confined his treatment to high frequencies since this corresponds to the Wien limit where he could treat them as independent. Firstly, he showed that the entropy change due a fluctuation process where N independent particles moving in a space of volume V_0 become confined to a sub-volume V is given by:

$$S - S_0 = kN \ln(V / V_0) . \tag{A.5}$$

Secondly, he compared this result with the entropy change from the Wien radiation formula for two systems of energy E occupying volumes V and V_0 of space:

$$S - S_0 = k(E/h\nu)\ln(V\,/\,V_0)\,. \tag{A.6}$$

It can be seen by comparing Equations (A.5) and (A.6) that the energy E of the heat radiation can be identified with N independent spatially localized quanta of magnitude $h\nu$, $E = Nh\nu$.

Issues in Bohr's Complementarity

Entanglement is responsible for the circumstance that "an unambiguous definition of the state of the system is naturally no longer possible." For Bohr, the "claim of causality" had meant a description in which energy and momentum are conserved. However, given Bohr's central role for the understanding of measurement, let us take a more careful look at a variety Bohr's statements regarding causality, closed and open systems, and other aspects of measurement from his statements in talks, papers and letters over the years. Though Bohr rarely invokes the "reduction of the wave-packet", he appears to relate this feature to the statistical nature of quantum mechanics in his 1927 Como lecture where had introduced complementarity [398, p. 94],

See the print edition of The Quantum Measurement Problem for quotation.

However, some thirty years later, Bohr is more explicit in his view that reduction of the wave packet is a necessary way of describing the situation due to the indivisibility of the quantum. In a letter to Pauli in 1955, Bohr explains [213, p. 568],

See the print edition of The Quantum Measurement Problem for quotation.

Note Bohr's emphasis here on "closed phenomena". In this passage, does Bohr imply causal behavior when he stresses "well-defined applications only to closed phenomena"? This would be supported by Bohr's use of "strictly speaking" when describing observation as a causal process in the following remark [214, p. 89]:

> *As all measurements thus concern bodies sufficiently heavy to permit the quantum to be neglected in their description, there is, strictly speaking, no new observational problem in atomic physics. The amplification of atomic effects, which makes it possible to base the account on measurable quantities and which gives the phenomena a peculiar closed character, only emphasizes the irreversibility characteristic of the very concept of observation. While, within the frame of classical physics, there is no difference in principle between the description of the measuring instruments and the objects under investigation, the situation is essentially different when we study quantum phenomena, since the quantum of action imposes restrictions on the description of the state of the systems by means of space-time coordinates and momentum-energy quantities.*

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This is consistent with the possibility that Bohr believes that closed systems are causal, but because the interaction required during measurement is uncontrollable the results have an apparent non-deterministic appearance. No rigorous theoretical argument appears to be given by Bohr to justify fundamental nondeterminism versus unknowable determinism because of the necessity of measurement in forming what we know. Deterministic evolution is consistent with the requirement of loss of phase that would result due to an uncontrollable interaction, as Bohr explains in the following, [398, p. 62]:

See the print edition of The Quantum Measurement Problem for quotation.

In a letter from Bohr to Pauli [398, p. 193], this concept is reaffirmed for which Bohr argued that the loss of phase that results from the measurement process limits the application of the superposition principle. It is well known that the loss of phase or a mixture also results in standard decoherence theory using deterministic evolution via Schrödinger's equation when considering system-measuring device interaction.

Does Bohr believe that the use of probability is fundamental or is being used because the underlying theory is deterministic but of such complexity that the use of probability is a good approximation? Consider Bohr's statement [494, p. 343],

See the print edition of The Quantum Measurement Problem for quotation.

Bohr claims that the use of probability is not to account for mechanical systems of great complexity. But why should quantum physics not be able to capture such intricacies? Bohr appears to believe that discontinuities were real and that nondeterminism was physical, but only when "open to direct observation" [439, p. 498],

> *Still it would appear that the essence of the theory may be expressed through the postulate that any atomic process open to direct observation involves an essential element of discontinuity or rather individuality completely foreign to the classical ideas and symbolized by Planck's quantum of action. This postulate at once implies resignation as regards the causal space-time coordination of atomic phenomena.*

The following argument appears to allow the possibility that closed systems obey Schrödinger's equation but are fundamentally unknowable due to interaction during measurement in order to observe such systems [213][Kindle Locations 5430-5432],

See the print edition of The Quantum Measurement Problem for quotation.

Similar arguments by Bohr center on the inability to determine the initial state of a system without affecting it due to the interaction with an external measuring device [310]:

> *The conception of a stationary state involves, strictly speaking, the exclusion of all interactions with individuals not belonging to the system. The fact that such a closed system is associated with a particular energy value, may be considered as an immediate expression for the claim of causality contained in the theorem of conservation of energy*.
>
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This is an argument that again is consistent with causal or deterministic evolution for closed systems. Bohr's arguments appear to only require the condition of deterministic but unknowable evolution for closed systems. This survey of Bohr's statements therefore reveals a more nuanced view of how he may have viewed causality and other aspects of measurement in closed and open systems.

Details from The Quantum Triumvirate

The violation of Bell's theorems by quantum entangled particles can be illustrated in a very direct way by considering, instead of two nonlocal particles, three nonlocal two-level particles sharing entanglement via the special properties of the Greenberger-Horne-Zeilinger (GHZ) state [568], which in the z-basis takes the form:

$$|GHZ\rangle = \frac{1}{\sqrt{2}}\left(|1_z\rangle_A|1_z\rangle_B|1_z\rangle_C - |-1_z\rangle_A|-1_z\rangle_B|-1_z\rangle_C\right).$$

These particles might be at the locations of spatially separated observers A, B, and C (i.e., Alice, Bob and Charlie) who each carry out measurements on the particle at their location using either Pauli matrix σ_x or σ_y , with the results being either +1 or -1. Following the description in Chapter 2 for treating polarized photons in terms of qubit states, these two measurements can be thought of as either representing position measurements of localized photon wave-packets (i.e., σ_x) or their counterparts after being sent through beam splitters (i.e. σ_y) [626]. A direct calculation using σ_x, σ_y and $|\mathrm{GHZ}\rangle$ finds that the results of measurements by A, B and C in the GHZ state always obey the following predictions of quantum theory:

$$\begin{aligned}
\{\sigma_{Ax}\}\{\sigma_{Bx}\}\{\sigma_{Cx}\} &= -1\\
\{\sigma_{Ax}\}\{\sigma_{By}\}\{\sigma_{Cy}\} &= 1\\
\{\sigma_{Ay}\}\{\sigma_{Bx}\}\{\sigma_{Cy}\} &= 1\\
\{\sigma_{Ay}\}\{\sigma_{By}\}\{\sigma_{Cx}\} &= 1
\end{aligned}$$

where $\{\sigma_{Ax}\}$ represents the result of measurement of σ_x by A, etc. Note the startling implication that the measurements of A, B, and C cannot be reproduced by local hidden variables as long as they don't conspire beforehand. This can be seen by multiplying the left-hand sides of these equations with the result $\{\sigma_{Ax}\}^2\{\sigma_{Ay}\}^2\{\sigma_{Bx}\}^2\{\sigma_{By}\}^2\{\sigma_{Cx}\}^2\{\sigma_{Cy}\}^2$, which equals just +1 since each local variable must have a definite value of either +1 or -1 by EPR's argument. However, the product of the right-hand sides is -1, giving a contradiction with quantum theory. Therefore, there cannot be definite values for any of the variables $\{\sigma_{Ax}\}$, $\{\sigma_{Ay}\}, \{\sigma_{Bx}\}, \{\sigma_{By}\}, \{\sigma_{Cx}\}, \{\sigma_{Cy}\}$ before the measurement takes place even though they are all locally measureable. This result requires us to reject local determinism.

Kraus Operators

In the 1970-1980s, von Neumann's formalism of measurement was generalized from projection operators to POVM's (Positive Operator Value Measurement) [627] [49] in terms of Kraus operators M_i [14, p. 91]

$$|\psi\rangle \longrightarrow \frac{M_i|\psi\rangle}{\||M_i||\psi\rangle\|}$$

where completeness requires $\sum_i M_i^\dagger M_i = I$. For a mixed state density operator,

$$\rho \longrightarrow \frac{M_i\rho M_i^\dagger}{\mathrm{Tr}\left(M_i\rho M_i^\dagger\right)}.$$

The measurement operators M_i represent properties of the measuring device and describe its effect on the system when the i-th outcome is registered. These generalized measurements are often viewed formally in terms of a projected measurement on an ancilla Hilbert space which is unitarily coupled to the system as justified by Naimark's Theorem [46] [628]. POVM's have been particularly useful for work in quantum information as realized in quantum optical and atomic systems. Within this framework, it is also straightforward to include the effects of disorder or information loss by averaging over subsets of Kraus operators, i.e., decoherence. In this way, a variety of extensions to the formalism of measurement have been developed that enable applications to a variety of phenomena including continuously monitored quantum systems, weak measurements, partial measurements, quantum trajectories, quantum filtering, feedback control, quantum jumps and stochastic master equations [186].

Quantum Trajectories and Jumps

The term *quantum trajectory* was introduced by Carmichael [184] for the stochastic evolution of a quantum state, and this been particularly useful for various types of photodetection including homodyne detection. This allows consideration of a sequence of measurements with the time evolution of the state conditioned on the results. The limit where the system is subjected to continuous measurement can then obey a stochastic version of the Schrödinger equation. These developments were motivated in part by the refining of experimental techniques to the point of enabling quantum-limited measurements on individual quantum systems. One of the first demonstrations of the power of these methods was the observation and prediction of quantum jumps of a single trapped ion [629] [183] [630]. These modern refinements brought a realization of the early concepts of the quantum founders from the 1920-30s, allowing the possibility to actually monitor the reduction of the wave function (i.e., Schrödinger's *damned jumping)* by the measurement process on an oscilloscope screen. These experiments also reveal that quantum jumps depend on the way the atom is monitored and what they describe is the state of the atom conditioned on the local state of the measuring device. The experimenter's decision to switch from observing circular rather than linear polarization results in the photon being emitted all at once as a quantum jump rather than leaking out in a diffusive way. This is precisely Bohr's complementarity [631]. These generalizations began within von Neumann's framework but now allowing analysis that goes far beyond basic projection operators as required by the myriad of contemporary tools for probing measurement. Despite the usefulness of these later developments for measurement techniques, they still do not bring us any closer to resolving the physics of the measurement problem. As discussed in Chapter 4, the status of the measurement problem remains the same as it did in von Neumann's day except for the deeper understanding of entanglement via the Bell inequality experiments.