Theory and Classification of Measurement Operations

Introduction

There exist numerous categories of measurement operations [656] [186] that have been considered in the literature. Whether or not such operations are possible in terms of actual measurement is unknown, and the successful resolution of the measurement problem is expected to provide further insight into this problem. In this section, some of these classes will be reviewed. It is sometimes found that different authors use the same term for different effects. These will be differentiated when found.

A projection operator P is a linear idempotent transformation, i.e., one for which $P^2 = P$. A projection operator has the property that repeated application of P to $P|\psi\rangle$ does not alter $P|\psi\rangle$. Note that all rank 1 operations of the form $|\psi\rangle\langle\psi|$ are projection operators, but a projection operator need not be rank 1.

A POVM of N possible outcomes is specified by a set of positive self-adjoint (Hermitian for finite dimensional matrices) operators of the form $E_k = M_k{}^\dagger M_k$, $k \in \Omega = \{1, \cdots, N\}$ with the properties

$$\sum E_k = I,$$

where $\langle\psi|E_k|\psi\rangle \geq 0 \ \forall \ |\psi\rangle \in \mathcal{H}_A, E_k \in \mathcal{B}(\mathcal{H})$, and $\mathcal{B}(\mathcal{H})$ denotes the set of bounded linear operators on the Hilbert space $\mathcal{H}$. When the k th result occurs, the density operator ϱ is transformed according to

$$\mathcal{E}_k(\varrho) = \frac{M_k \varrho M_k{}^\dagger}{p_k} \tag{7.1}$$

with probability $p_k = \text{Tr}[M_k \varrho M_k{}^\dagger] = \text{Tr}[\varrho E_k]$. We define POVM observables via a set of positive operators $\{E_k\}$ satisfying $\sum E_k = I$, $E_k \in \mathcal{B}(\mathcal{H})$. Aside from the possibility of non-Hermitian decompositions M_k, $E_k = M_k{}^\dagger M_k$ one can always find a decomposition $E_k = A_k{}^2$, where A_k is any square root of E_k. Note that A_k are not necessarily unique and the map $\mathcal{E}(\varrho)$ may not be unique for a given set of operators E_k. When the A_k are further constrained to be positive semi-definite, A_k can also be shown to be unique and will be denoted $A_k = E_k{}^{1/2}$.

We note that there exists a more general representation of $\mathcal{E}_k(\varrho)$ for which $\mathcal{E}_k(\varrho) = \sum_i p_k \mathcal{E}_{k,i}(\varrho)$ that are defined as *inefficient* in [186, p. 32]. In such cases a pure state is generally transformed to a mixed state for a given outcome. Although this is possible mathematically, the authors are not aware of any physical measurement that has been shown to be necessarily caused by an inefficient non-unitary action, and hence the question of whether or not physical measurement operations allow such measurements is an open question. On the other hand, experimental evidence does support the form of Equation (7.1). Henceforth, all measurements will be assumed to be of the form of Equation (7.1). The density matrix that is averaged over all

measurement results is often referred to as Kraus decomposition or operator-sum form:

$$T(\varrho) = \sum M_k \varrho M_k^\dagger. \tag{7.2}$$

Note that for a given set of POVM elements E_k, the transformation of the density matrix is not unique, until a particular decomposition is considered of the E_k elements that are to comprise the operators in the transformation of the density matrix. Hence there can be different implementations of a given set of POVM elements E_k. Projection operators P_k that have a non-trivial degenerate subspace transform the density matrix according to an Ansatz by Lüders that corrected a deficiency in von Neumann's original treatment. A Lüders [657] transformation occurs when the density matrix transforms according to:

$$\mathcal{E}_k^L(\varrho) \equiv \frac{P_k\, \varrho P_k}{P(k)}. \tag{7.3}$$

A general transformation of ϱ may or may not be a Lüder's transformation. We denote the average density matrix given a Lüder's instrument as $T^L(\varrho)$. Consider the following which can be used to consider a von Neumann Hermitian observable A (which is not necessarily a positive matrix) as a POVM. A Hermitian matrix has a spectral decomposition. $A = \sum_i \lambda_i P_i$ where the P_i are projection operators and λ_i are the unique eigenvalues of A. The probability of the k th outcome of the observable A is given by $\text{Tr}(\varrho A_k)$. A POVM with elements $\{A_k\}$ would yield the correct probabilities of the observable. However, not only does the probability of an observable need to coincide with von Neumann's postulate, but the state transformation needs to coincide with the projection onto the eigenstates of an observable. The POVM defined via Lüder's instrument of Equation (7.3) suffices to implement a von Neumann Hermitian observable.

It has been noted previously that in defining POVM positive operators $\{E_k\}$ satisfying $\sum E_k = I$, $E_k \in \mathcal{B}(\mathcal{H})$, that the map $\mathcal{E}(\varrho)$ may not be unique for a given set of operators E_k. When applying the decomposition $E_k = E_k^{1/2} E_k^{1/2}$, we will refer to the resulting implementation as a generalized Lüder's instrument. The density matrix when the k th outcome occurs is given by

$$\mathcal{E}_k^L(\varrho) = \frac{E_k^{1/2}\, \varrho E_k^{1/2}}{P(k)}, \tag{7.4}$$

where $P(k)$ is the probability of the k th outcome. An adjoint map of $\mathcal{E}_k(\varrho)$ that maps observables, denoted $\mathcal{E}_k^*(A) \colon \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$, $A \in \mathcal{B}(\mathcal{H})$, can be defined via the representation for which the operator is transformed and the state ϱ is constant. In analogy to the unitary case for which the $\mathcal{E}_k^*(A)$ is the Heisenberg representation and $\mathcal{E}_k(\varrho)$ the Schrödinger representation, the probability of measurement must be the same, which is given by the duality requirement that $\text{Tr}(A_k \mathcal{E}_k(\varrho)) = \text{Tr}(\mathcal{E}_k^*(A_k)\varrho)$.

A POVM observable $A = \{A_k\}$ will be said to be commutative if $[A_i, A_j] = 0$, $i \neq$

$j \in \Omega_A$ where Ω_A is the outcome space for A. Two POVMs $A = \{A_k\}$, $B = \{B_k\}$ with outcomes in Ω_A and Ω_B respectively are said to commute if $[A_i, B_j] = 0$, for all $i, j \in \Omega_A \times \Omega_B$, where $[A, B]$ denotes the commutator between A and B. A POVM observable A has rank 1 if $\mathcal{R}(A_i) = 1, \forall i$, where $\mathcal{R}(B)$ denotes the rank of the matrix B.

A joint measurement in its most general form is a single measurement $C = \{C_{i,j}\}$ that has outcomes in the product space $i, j \in \Omega_A \times \Omega_B$. Note that sequential measurements such as A followed by B is a joint measurement, but not all joint measurements need be sequential measurements.

Given a joint measurement C, the marginal observables are computed as i.e. $A = \sum_j C_{i,j}$, and $B = \sum_i C_{i,j}$. Two POVM observables A and B are said to be jointly measurable if there exists a measurement C on $\Omega_A \times \Omega_B$ that returns A and B as marginal POVM observables. POVM observables A and B that commute, are also jointly measurable [229].