Completely Positive Maps

Consider a Hilbert space $\mathcal{H}_A$ and a linear map $\mathcal{E}(\varrho)$ where ϱ is a density operator on $\mathcal{H}_A$. If $\mathcal{E}(\varrho)$ is positive for all density matrices ϱ, then $\mathcal{E}$ is defined as a positive map.

Suppose a density matrix ρ_{AB} of a bipartite system A, B with Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ is acted upon locally on A, i.e. via an operation $\hat{\rho}_{AB} = (\mathcal{E} \otimes I)\rho_{AB}$. A positive map $\mathcal{E}$ defined on $\mathcal{H}_A$ is a completely positive (CP) map for which $\hat{\rho}_{AB}$ is also positive on $\mathcal{H}_A \otimes \mathcal{H}_B$ for all ρ_{AB}. A trace preserving linear map has a Kraus sum-operator representation if and only if it is completely positive (CP). If a trace preserving linear map $\mathcal{E}(\varrho)$ is also completely positive, then $\mathcal{E}$ is a quantum channel.

Note that when the initial state is a tensor product $\rho_{AB} = \varrho_A \otimes \varrho_B$, it is seen that $(\mathcal{E} \otimes I)\rho_{AB} = \mathcal{E}(\varrho_A) \otimes \varrho_B$ is always positive since $\mathcal{E}(\varrho_A)$ is positive. Additionally, if one considers a unitary operation $U\rho_{AB}U'$, it is found that if one restricts initial states to be tensor product initial states, such restricted initial state maps also have a Kraus representation for the reduced dynamics corresponding to Markovian dynamics. One might also consider initially correlated states. In the presence of initial correlations, such a map may not exist and the dynamics may not be Markovian. However, state dependent Kraus operators always can be found as a function of the initial condition and the unitary evolution operator [649]. Further issues of non-Markovian evolution have been explored in [650] [651].

For certain correlated initial states, non-CP maps may be physically relevant, but they found additional restrictions were needed [652]. The authors state:

See the print edition of The Quantum Measurement Problem for quotation.

It has also been shown in [653] that there exist linear density maps that describe the evolution of a subsystem that is initially entangled with another system for which the linear map is not completely positive. Hence it appears that certain relevant physical open system maps can be linear, and may not be CP.

A point argued in [647] is that the general form of quantum state evolution, whether unitary or measurement, is completely positive. The argument is essentially that if one allows entangled states on $\mathcal{H}_A \otimes \mathcal{H}_B$ and a non-CP local operation on B, that there would be cases for which the overall state would be not positive, which would contradict the basic tenants of positive probabilities upon which von Neumann theory, as well as probability theory, is predicated. This is true, if all states are allowed to occur in a Hilbert space.

On the other hand, a characteristic of measurement is that macroscopic superpositions are not always seen. The measurement problem requires one to explain why the entangled states that are predicted under unitary evolution are not occurring. It may be a valid induction that entangled states can occur, but at least from what has been established at this point, it is not a valid deduction based on the evidence of the characteristics of the problem for which entangled superpositions are not seen. One might argue that such states could occur, but that they are impeded by some physical reason such as Penrose's gravitational reduction and such impedance reduces such states to a product state. That is certainly one possibility. Another logical possibility that must be considered in a deductive investigation is that certain states can never occur. And it has been shown in [648] that if one assumes linear density operator evolution and that density matrices are mapped one-to-one and onto density matrices, then the resulting evolution is unitary. Hence it seems the inability of certain states to ever occur is a feature that is directly related to von Neumann measurement and the existence of a process that is different from unitary evolution. Due to the fact that measurement is not a one-to-one and onto process, certain states are excluded from occurring by the formalism itself, and hence the issue of whether or not all operations are CP, particularly those in the process of measurement, does not appear to have been properly addressed, as of yet.

There still exist some questions in the literature as noted in [652]:

> *Despite much recent attention and progress, the situation for more general initial conditions (e.g., for families of thermal states) is not well understood. Is completely positivity indispensable? Is there a consistent framework for the dynamics arising from general initial conditions?*

It had been claimed [654] that the equation for wave function collapse is necessarily CP under the assumption of linear density operator and Markov evolution. If this were the case, then CP would certainly form a natural addition to linear density operator and the Lindblad CP form could be taken as a deductive assumption for the general form of quantum evolution. However, the claim in [654] was shown by counterexample to be false by Diosi [655] who exhibited a Markov process that is not CP.

As in the case of linear density operator, the case of CP evolution at this time does appear to be a compelling argument when restricted to theories in which all superposition states can occur. Until such time as it is ruled out, CP maps appear to be a potential candidate to describe quantum evolution in the case of theories that strictly rule out certain superposition states from physically ever occurring. In the case of the latter, it is known that there are cases where superpositions of certain eigenstates never occur, and the imposition of superselection rules is required. For example, pure superposition states of different charge are prohibited. In such cases, the possibility of non-CP maps should also be considered until such time as the matter is more thoroughly understood, and before reaching firm conclusions in a deductive investigation that could very well rule out a non-CP theory that could turn out to be the correct theory.